Multiplicative chaos in random matrix theory and related fields

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The GUE eigenvalue counting function.

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3. Consider the stochastic process $\frac{e^{\gamma V_N(x)}}{\mathbb{E}e^{\gamma V_N(x)}}$ for $x \in (-1, 1)$ and $\gamma \in \mathbb{R}$. 

Moments converge as $N \to \infty$:

Theorem (Charlier 2017)

Let $x_1, \ldots, x_k \in (-1, 1)$ be fixed and distinct. Then

$$\lim_{N \to \infty} \mathbb{E} \prod_{j=1}^{k} e^{\gamma V_N(x_j)} = \prod_{1 \leq p < q \leq k} \left| 1 - x_p x_q + \sqrt{1 - x_p^2} \sqrt{1 - x_q^2} \right|^\gamma \frac{\pi}{2}.$$
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- Is there a process with such moments? Does $\frac{e^{\gamma V_N(x)}}{E e^{\gamma V_N(x)}}$ converge to it? What would this say about the GUE?
The limiting process – heuristics

- For $(Y_k)_{k=1}^{\infty}$ i.i.d. standard Gaussians and $(U_j)_{j=0}^{\infty}$ Chebyshev polynomials of the second kind, let (formally):

\[
X(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{Y_k}{\sqrt{k}} U_{k-1}(x) \sqrt{1 - x^2}.
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- Covariance structure (formally): for \(x, y \in (-1, 1)\)

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\mathbb{E}X(x)X(y) = \frac{1}{2\pi^2} \log \frac{1 - xy + \sqrt{1 - x^2} \sqrt{1 - y^2}}{|x - y|}.
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- For \(\mu_{\gamma}(x) = e^{\gamma X(x)} - \frac{\gamma^2}{2} \mathbb{E} X(x)^2\) (formally)

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\mathbb{E} \prod_{j=1}^{k} \mu_{\gamma}(x_j) = \prod_{1 \leq p < q \leq k} \left| \frac{1 - x_p x_q + \sqrt{1 - x_p^2} \sqrt{1 - x_q^2}}{x_p - x_q} \right|^2 \frac{\gamma^2}{2\pi^2}.
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• For each \(x\), the sum defining \(X(x)\) diverges almost surely and \(\mathbb{E}X(x)^2 = \infty\). What does \(\mu_\gamma\) mean? 😐
Gaussian multiplicative chaos – rigorous construction

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- Can check that for nice test functions $f$ and for $-\sqrt{2}/\pi < \gamma < \sqrt{2}/\pi$ the limits exist as we’re dealing with $L^2$-bounded martingales (actually OK for $-2\pi < \gamma < 2\pi$ – $L^p$-bounded martingale).
- This procedure defines random measures/distributions. These are the objects we are after – correlation kernels agree with the limiting GUE-moments.
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Real Gaussian multiplicative chaos – the general picture

- A centered **log-correlated Gaussian field** $G(x)$ is (formally) a Gaussian process on $\mathbb{R}^d$ with covariance

$$C(x, y) := \mathbb{E} G(x)G(y) = -\log |x - y| + \text{continuous}$$
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**Theorem (Kahane 1985,...)**

*For nice enough $C(x, y)$, as $N \to \infty$:

- $e^{\gamma G_N(x) - \frac{\gamma^2}{2} \mathbb{E} G_N(x)^2} dx$ converges to a non-trivial random measure $M_\gamma$ for $-\sqrt{2d} < \gamma < \sqrt{2d}$. For $|\gamma| \geq \sqrt{2d}$, the limit is zero.

- For $|\gamma| < \sqrt{2d}$, $M_\gamma$ lives on the random set of points (of dimension $d - \frac{\gamma^2}{2}$)

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- Interpretation: GMC $\to$ level sets. $\max_x G_N(x) \sim \sqrt{2d} \mathbb{E} G_N(x)^2$. 
GMC in other fields of mathematics

- Initial motivation for GMC (Mandelbrot, Kahane): statistical description of turbulence – $M_\gamma =$ energy dissipation density.

- $M_\gamma$ can be seen as the (unnormalized) Gibbs measure of a random energy model with (logarithmic) correlations.

- Connections to RMT and number theory suggested by Fyodorov and Keating.

- Connections to random planar curves (SLE) through conformal welding. (Sheffield; Astala et al.)

- Connections to 2d quantum gravity and random planar maps. $M_\gamma$ (for suitable $\gamma$ and the 2d GFF) plays a role in constructing scaling limits of random planar maps. (Duplantier, Miller, and Sheffield)

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**Theorem (Claeys, Fahs, Lambert, W 2018)**

Let $\gamma \in (-2\pi, 2\pi)$ and $f \in C_c((-1, 1))$. Then as $N \to \infty$

$$\int_{-1}^{1} f(x) \frac{e^{\gamma V_N(x)}}{\mathbb{E}e^{\gamma V_N(x)}} \, dx \overset{d}{\to} \int_{-1}^{1} f(x) \mu_\gamma(dx).$$
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For any $\epsilon, \delta > 0$ fixed, $\lambda_1 \leq \ldots \leq \lambda_N$ as before, and $\rho_k$ the classical locations of the eigenvalues:

$$\lim_{N \to \infty} P\left( \frac{1}{\pi} - \epsilon \leq \sup_{\delta N \leq k \leq (1 - \delta)N} \left| \lambda_k - \rho_k \right| \leq \frac{1}{\pi} + \epsilon \right) = 1.$$
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The Riemann zeta function

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}, \quad \Re(s) > 1. \]
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- Has a meromorphic continuation to \( \mathbb{C} \), with a single pole at \( s = 1 \). Behavior of \( \zeta(\frac{1}{2} + it) \) is of fundamental importance in analytic number theory (distribution of primes etc).
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**Theorem (Ingham 1926, Bettin 2010)**

Let \( \omega \) be uniformly distributed on \([0, 1]\) and \( x, y \in \mathbb{R} \) be fixed. As \( T \to \infty \)

\[
\mathbb{E} \zeta \left( \frac{1}{2} + i x + i \omega T \right) \zeta \left( \frac{1}{2} + i y + i \omega T \right) = \zeta(1 + i(x - y)) + \frac{\zeta(1-i(x-y))}{1-i(x-y)} \left( \frac{T}{2\pi} \right)^{-i(x-y)} + O(T^{-1/12}).
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- Has a meromorphic continuation to \( \mathbb{C} \), with a single pole at \( s = 1 \).
- Behavior of \( \zeta(\frac{1}{2} + it) \) is of fundamental importance in analytic number theory (distribution of primes etc).
- Statistical behavior of \( \zeta(\frac{1}{2} + it) \) expected to be similar to characteristic polynomials of random matrices, but little is known rigorously.

**Theorem (Ingham 1926, Bettin 2010)**

Let \( \omega \) be uniformly distributed on \([0, 1]\) and \( x, y \in \mathbb{R} \) be fixed. As \( T \to \infty \)

\[
\mathbb{E} \zeta \left( \frac{1}{2} + ix + i\omega T \right) \zeta \left( \frac{1}{2} + iy + i\omega T \right)
= \zeta(1 + i(x - y)) + \frac{\zeta(1 - i(x - y))}{1 - i(x - y)} \left( \frac{T}{2\pi} \right)^{-i(x-y)} + \mathcal{O}(T^{-1/12}).
\]

- Does \( \lim_{T \to \infty} \zeta \left( \frac{1}{2} + ix + i\omega T \right) \) exist? ...
Multiplicative chaos and the Riemann zeta

Theorem (Saksman, W (2016))

• For any \( f \in C_c^\infty(\mathbb{R}, \mathbb{C}) \),

\[
\int \zeta \left( \frac{1}{2} + i \omega T + ix \right) f(x) dx \xrightarrow{d} \langle \xi, f \rangle \quad \text{as} \quad T \to \infty
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  \]

- \( \xi = \prod_{k=1}^\infty (1 - p_k^{-\frac{1}{2} - ix} e^{i\theta_k})^{-1} \overset{d}{=} e^{E\nu} \), where \( \theta_k \) i.i.d. and uniform on \([0, 2\pi]\), \( E \) is a random smooth function, and \( \nu \) is a complex GMC distribution.
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- On a suitable mesoscopic scale, \( \zeta \left( \frac{1}{2} + i\omega T + ix \right) \) is asymptotically proportional to the characteristic polynomial of a Haar distributed random unitary matrix.
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• For any \( f \in C_\infty_c(\mathbb{R}, \mathbb{C}) \),

\[
\int \zeta \left( \frac{1}{2} + i\omega T + ix \right) f(x) dx \xrightarrow{d} \langle \xi, f \rangle \quad \text{as} \quad T \to \infty
\]

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• \( \xi = \prod_{k=1}^{\infty} (1 - p_k^{-\frac{1}{2}} e^{i \theta_k})^{-1} \overset{d}{=} e^{E \upsilon}, \) where \( \theta_k \) i.i.d. and uniform on \([0, 2\pi]\), \( E \) is a random smooth function, and \( \upsilon \) is a complex GMC distribution.

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• Proof philosophy similar to GUE. Methods fairly basic number theory.
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  \]

- \( \xi = \prod_{k=1}^\infty (1 - p_k^{-\frac{1}{2}} - ix e^{i\theta_k})^{-1} \xrightarrow{d} e^{E_{\mathcal{V}}} \), where \( \theta_k \) i.i.d. and uniform on \([0, 2\pi]\), \( E \) is a random smooth function, and \( \mathcal{V} \) is a complex GMC distribution.

- On a suitable mesoscopic scale, \( \zeta \left( \frac{1}{2} + i\omega T + ix \right) \) is asymptotically proportional to the characteristic polynomial of a Haar distributed random unitary matrix.

- (Stronger results) conjectured by Fyodorov and Keating.
- Proof philosophy similar to GUE. Methods fairly basic number theory.
- Geometric interpretation? Interesting results about \( \max \text{Re/Im} \log \zeta \left( \frac{1}{2} + ix + i\omega T \right) \) exist: see Najnudel; Arguin et al.
The critical Ising model

• Let $U$ be a bounded simply connected domain in $\mathbb{C}$ and $U_\delta$ a lattice approximation of $U$ of mesh $\delta > 0$. 
The critical Ising model

- Let $U$ be a bounded simply connected domain in $\mathbb{C}$ and $U_\delta$ a lattice approximation of $U$ of mesh $\delta > 0$.
- Let $(\sigma_\delta(a))_{a \in U_\delta}$ be a spin configuration distributed according to the critical Ising model on $U_\delta$ with $+$ b.c. Extend $\sigma_\delta$ to $U$. 

Theorem (Chelkak, Hongler, and Izyurov 2015)

Let $\psi$ be any conformal bijection from $U$ to the upper half plane and $C$ a suitable constant. Then for $x_1, \ldots, x_k \in U$ fixed and distinct,

$$\lim_{\delta \to 0} \frac{\psi'_{\psi(x_j)} \psi_{\psi(x_j)}^{1/4}}{\sum_{\mu \in \{-1,1\}^k} \prod_{1 \leq p < q \leq k} |\psi_{\psi(x_p)} - \psi_{\psi(x_q)}|_{2 \text{Im}(\psi(x_j))}^{1/4} \prod_{j=1}^k \sigma_\delta(x_j)^2} = C$$
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$$

$$
= C^k \prod_{j=1}^{k} \left( \frac{|\psi'(x_j)|}{2 \text{Im}(\psi(x_j))} \right)^{1/4} \sum_{\mu \in \{-1,1\}^k} \prod_{1 \leq p < q \leq k} \left| \frac{\psi(x_p) - \psi(x_q)}{\psi(x_p) - \psi(x_q)} \right|^\frac{\mu_p \mu_q}{2}
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$$

- If $\sigma_\delta$ and $\widetilde{\sigma_\delta}$ are independent copies, does $x \mapsto \delta^{-1/4} \sigma_\delta(x) \widetilde{\sigma_\delta}(x)$ converge to some process (known that $\delta^{-1/8} \sigma_\delta(x)$ does)? ...
The critical Ising model

Theorem (Junnila, Saksman, W 2018)

Let $\sigma_\delta$ and $\tilde{\sigma}_\delta$ be independent copies of the Ising spin field. Then for any $f \in C_c^\infty(U)$, as $\delta \to 0$

$$
\delta^{-1/4} \int_U f(x) \sigma_\delta(x) \tilde{\sigma}_\delta(x) \, dx \xrightarrow{d} \int \mathcal{C} \left( \frac{\left| \psi'(x) \right|}{2 \text{Im} \psi(x)} \right) : \cos \text{GFF}(x) : f(x) \, dx.
$$
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- Known well in the physics literature – bosonization of the Ising model. See also work of Dubédat.
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- Proof is through method of moments: Chelkak, Hongler, and Izyurov + some rather easy bounds near the diagonal.
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- Geometric interpretation?
The GFF and $\cos(\text{GFF})$: images

![GFF Image](image1)

![cos(GFF) Image](image2)