On foliations related to the center of mass in General Relativity

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Consider initial data \((M^3, g, K, \mu, J)\) which are “optimally” asymptotically flat:

\[
M^3 \approx \mathbb{R}^3 \setminus \text{ball } \ni \vec{x}
\]

\[
g_{ij} = \delta_{ij} + \mathcal{O}_2(r^{-\frac{1}{2} - \varepsilon})
\]

\[
K_{ij} = \mathcal{O}_1(r^{-\frac{3}{2} - \varepsilon})
\]

\[
\mu, J = \mathcal{O}_0(r^{-3 - \varepsilon})
\]

for some \(\varepsilon > 0\) and \(r = |\vec{x}| \to \infty\).
Expectations of a notion of center of mass

- Transforms like a point particle in Special Relativity under change of observer:
  \[ t \overset{\sim}{\rightarrow} \tilde{t} \]  
  \[ t = 0 \overset{\sim}{\rightarrow} \tilde{t} = 0 \]  
  \[ t = 1 \overset{\sim}{\rightarrow} \tilde{t} = 1 \]

- Equivariant transformation behavior under asymptotic boosts.

- Equivariant transformation under spatial translations and rotations.
- Point particle-like evolution under Einstein evolution equations:

  \[
  \frac{d}{dt}(E \vec{z}) = \vec{P}
  \]

  (ADM-energy \( E \), ADM-momentum \( \vec{P} \))

- Newtonian limit \( c \rightarrow \infty \) of \( \vec{z}(c) \) recovers Newtonian center of mass of \( \vec{z} \) limiting Newtonian isolated system.
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- Newtonian limit \( c \rightarrow \infty \) of \( \tilde{z}'(c) \) recovers Newtonian center of mass of \( \tilde{z}' \) limiting Newtonian isolated system
Status quo

Different definitions of center of mass in the literature:

- **Definition via Hamiltonian systems:**
  Regge–Teitelboim ’74, Beig–Ó Murchadha ’87.
  \(\leadsto\) does not transform equivariantly and does not converge in general

- **Asymptotic foliation definition by Huisken–Yau ’96.**
  \(\leadsto\) see below

- **Several others (Schoen, Corvino–Wu, Chen–Wang–Yau, . . . ).**
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Excursion: Isolated systems in Newtonian Gravity

Center of mass $\vec{z}_N \in \mathbb{R}^3$ of a mass density $\rho$ and mass $m_N = \int_{\mathbb{R}^3} \rho \, dV \neq 0$:

$$\vec{z}_N = \frac{1}{m_N} \int_{\mathbb{R}^3} \rho \, \vec{x} \, dV.$$ 

Can be reformulated: $U$ Newtonian potential with $U \to 0$ as $r \to \infty$:

$$\triangle U = 4\pi \rho.$$ 

If $m_N \neq 0$: equipotential sets $\Sigma_U$ foliate neighborhood of infinity. 

Recover $\vec{z}_N$ from

$$\vec{z}_N = \lim_{U \to 0} \int_{\Sigma_U} \vec{x} \, dA.$$
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Theorem (Huisken–Yau ’96; abstract CoM)

Let $(M^3, g)$ be an asymptotically spherically symmetric Riemannian manifold of mass $m > 0$. There exists an (almost) unique foliation of a neighborhood of infinity by stable spheres $\Sigma_H$ of constant mean curvature $H$ (CMC).

- Asymptotic condition: $g_{ij} = (1 + \frac{m}{2r})^4 \delta_{ij} + O(\frac{1}{r^2})$.
- Generalizations: Ye, Metzger, Metzger–Eichmair, Huang, Nerz, ...
Theorem (Huisken–Yau ’96; coordinate CoM)

Euclidean center $\vec{z}_H$ of $\Sigma_H$ and center of mass $\vec{z}_{HY}$:

$$\vec{z}_H := \int x dA, \quad \vec{z}_{HY} := \lim_{H \to 0} \vec{z}_H.$$
However:

Theorem (C.–Nerz '14)

Der center of mass \( \vec{z}_{HY} := \lim_{H \to 0} \vec{z}_H \) does not always converge under the assumptions of Huisken–Yau.

Explicit counterexample: graphical timeslice in Schwarzschild spacetime:

\[
T(\vec{x}) = \frac{\vec{a} \cdot \vec{x}}{r} + \sin(\ln r),
\]

\( \vec{a} \in \mathbb{R}^3, \vec{a} \neq 0 \)

Figure: Logarithmic plot.

- Reason: \( \vec{R} \vec{x} \notin L^1 \) in general, \( \vec{R} \) scalar curvature of \( g \).
- Same phenomenon in Newtonian setting by changing coordinates asymptotically if \( \rho \vec{x} \notin L^1 \).
However:

**Theorem (C.–Nerz ’14)**

*Der center of mass* $\vec{z}_{HY} := \lim_{H \to 0} \vec{z}_H$ *does not always converge under the assumptions of Huisken–Yau.*

**Figure:** Logarithmic plot.

- **Explicit counterexample:**
  - Graphical timeslice in Schwarzschild spacetime:
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    \]
    \[
    \vec{a} \in \mathbb{R}^3, \vec{a} \neq 0
    \]

- **Reason:** $\mathbb{R} \vec{x} \notin L^1$ in general, $\mathbb{R}$ scalar curvature of $g$.
- **Same phenomenon in Newtonian setting by changing coordinates asymptotically** if $\rho \vec{x} \notin L^1$. 
New development

Theorem (C.–Sakovich ’18)

Let $(\mathcal{M}^3, g, K, \mu, J)$ be initial data. Under optimal asymptotic flatness conditions and if the ADM-energy $E \neq 0$, there exists a unique foliation of a neighborhood of infinity by stable spheres $\Sigma_\mathcal{H}$ of constant spacetime mean curvature $\mathcal{H} = \sqrt{g(\mathcal{H}, \mathcal{H})}$ (STCMC).

Assuming $\mu \vec{x} \in L^1$, the euclidean center $\vec{z}_\mathcal{H}$ of $\Sigma_\mathcal{H}$ and the center of mass $\vec{z}$ satisfies:\n
$$
\vec{z}_\mathcal{H} := \int_{x^i(\Sigma_\mathcal{H})} \vec{x} \, dA, \quad \vec{z} := \lim_{\mathcal{H} \to 0} \vec{z}_\mathcal{H}.
$$

\textsuperscript{a}Under a weak additional decay assumption on $K$ which seem technical.
Coordinate STCMC-center of mass
New development.

**Theorem (C.–Sakovich ’18)**

- The STCMC-center of mass \( \vec{z} \) transforms equivariantly under the asymptotic Poincaré group (in coordinates), i.e. under boosts and spatial translations and rotations, as well as

- *point particle-like evolution under the Einstein evolution equations via*

\[
\frac{d}{dt} (E \vec{z}) = \vec{P}.
\]

- *The counterexample from [C.–Nerz ’14] has a well-defined STCMC-center of mass \( \vec{z} = \vec{0} \).*

New development...

- Have explicit formula for difference between $\vec{z}_{HY}$ and new $\vec{z}$ via BÓM–RT-formula (Huang, Nerz, ...).

- Agrees with Chen–Wang–Yau center of mass if initial data are asymptotically harmonic.

- Work in progress with Metzger: The extra weak additional decay assumption on $K$ is not necessary but can be replaced by choosing suitable center of mass coordinates.
Open question: Newtonian limit of center of mass...

**Theorem (C. ’11)**

Along each $c$-dependant family of static isolated systems that has a Newtonian limit as $c \to \infty$, one finds that

$$\vec{z}_{HY}(c) = \vec{z}_{BOM-RT}(c) = \vec{z}_{PN}(c) \to \vec{z}_N.$$  

Proof: Ehlers’ frame theory, differential geometry modelling, Kelvin transformation, weighted Sobolev space analysis, faster fall-off trick [C. ’11], localization of mass and center of mass via pseudo-Newtonian gravity.