Telegraph equation from the six-vertex model

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Telegraph equation

Resistance $R$  Inductance $L$  Capacitance $C$  Conductance $G$

Voltage $V$  Current $I$

$$\frac{\partial V}{\partial x}(x, t) = -L \cdot \frac{\partial I}{\partial t}(x, t) - R \cdot I(x, t)$$

$$\frac{\partial I}{\partial x}(x, t) = -C \cdot \frac{\partial V}{\partial t}(x, t) - G \cdot V(x, t)$$

or

$$V_{xx} - LC \cdot V_{tt} - (RC + GL) \cdot V_t - GR \cdot V = 0$$

Wave equation  Effect of losses
Six–vertex model

Square grid with $O$ in the vertices and $H$ on the edges.
Six–vertex model

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Finite/infinite domain.
Six–vertex model

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*Configurations:* possible matchings of all atoms inside domain into $H_2O$ molecules.

This is *square ice model*. Real-world ice has somewhat similar (although 3d) structure.
Six–vertex model

Square grid with $O$ in the vertices and $H$ on the edges.

Finite/infinite domain.

**Configurations**: possible matchings of all atoms inside domain into $H_2O$ molecules.

This is **square ice model**. Real-world ice has somewhat similar (although 3d) structure.

Also known as **six vertex model**.
Six–vertex model: Gibbs measures

\[ \begin{align*}
\text{Assign Gibbs weights} \\
\frac{a_1^{\#(a_1)} a_2^{\#(a_2)} b_1^{\#(b_1)} b_2^{\#(b_2)} c_1^{\#(c_1)} c_2^{\#(c_2)}}{Z(a_1, a_2, b_1, b_2, c_1, c_2)}
\end{align*} \]

[ Depends only on \( \frac{b_1 b_2}{a_1 a_2} \) and \( \frac{c_1 c_2}{a_1 a_2} \).]

Asymptotic properties of Gibbs measures?
Assign Gibbs weights

\[
\frac{\#(a_1) \#(a_2) \#(b_1) \#(b_2) \#(c_1) \#(c_2)}{Z(a_1, a_2, b_1, b_2, c_1, c_2)}
\]

[ Depends only on \( \frac{b_1 b_2}{a_1 a_2} \) and \( \frac{c_1 c_2}{a_1 a_2} \).]

*Asymptotic* properties of Gibbs measures?

Understanding is still very limited.
Question for today

Telegraph equation

\[ V_{xx} - V_{tt} - \alpha V_t - \beta V_x - \gamma V = 0 \]

Six–vertex model

What do they have in common?
Stochastic six–vertex model

An equivalent representation
Collection of paths on the plane
Stochastic six–vertex model

\[ \begin{array}{cccccc}
1 & 1 & b_1 & b_2 & 1 - b_1 & 1 - b_2 \\
\end{array} \]

\[ \begin{array}{cccccc}
\text{H} & \text{O} & \text{O} & \text{H} & \text{H} & \text{O} \\
\text{H} & \text{H} & \text{O} & \text{H} & \text{H} & \text{O} \\
\end{array} \]

Assumption:
(Gwa–Spohn–92)

\[ a_1 = a_2 = 1, \quad b_1 + c_1 = 1, \quad b_2 + c_2 = 1 \]

(Implies \( \Delta = \frac{a_1 a_2 + b_1 b_2 - c_1 c_2}{2 \sqrt{a_1 a_2 b_1 b_2}} \geq 1 \))
Stochastic six–vertex model

\[
\begin{pmatrix}
1 & 1 & b_1 & b_2 & 1 - b_1 & 1 - b_2 \\
\end{pmatrix}
\]

Take arbitrary **boundary conditions** in the quadrant
Stochastic six–vertex model

\[
\begin{array}{cccccc}
1 & 1 & b_1 & b_2 & 1 - b_1 & 1 - b_2 \\
\end{array}
\]

Proceed with \textit{sequential stochastic sampling}
Stochastic six–vertex model

\[ 1 \quad 1 \quad b_1 \quad b_2 \quad 1 - b_1 \quad 1 - b_2 \]

Proceed with *sequential stochastic sampling*

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & \cdots \\
\hline
1 & & & & & \\
2 & & & & & \\
3 & & & & & \\
4 & & & & & \\
5 & & & & & \\
\end{array}
\]

no choice
Stochastic six–vertex model

\[
\begin{array}{cccccc}
1 & 1 & b_1 & b_2 & 1 - b_1 & 1 - b_2 \\
\end{array}
\]

Proceed with sequential stochastic sampling

no choice
Stochastic six–vertex model

\begin{align*}
1 & & 1 & & b_1 & & b_2 & & 1 - b_1 & & 1 - b_2 \\
\end{align*}

Proceed with \textit{sequential stochastic sampling}
Stochastic six–vertex model

Proceed with sequential stochastic sampling
Stochastic six–vertex model

Proceed with sequential stochastic sampling

no choice
Stochastic six–vertex model

\[
\begin{bmatrix}
1 & 1 & b_1 & b_2 & 1 - b_1 & 1 - b_2 \\
\end{bmatrix}
\]

Proceed with \textit{sequential stochastic sampling}
Stochastic six–vertex model

\[
\begin{array}{cccccc}
1 & 1 & b_1 & b_2 & 1 - b_1 & 1 - b_2 \\
\end{array}
\]

Proceed with **sequential stochastic sampling**
Stochastic six–vertex model

\begin{align*}
\begin{array}{cccccccc}
1 & 1 & b_1 & b_2 & 1 - b_1 & 1 - b_2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
5 & & & & & & & \\
\end{array}
\end{align*}

Proceed with \textbf{sequential stochastic sampling}
Stochastic six–vertex model

1 1 \( b_1 \) \( b_2 \) 1 − \( b_1 \) 1 − \( b_2 \)

Proceed with \textbf{sequential stochastic sampling}
Stochastic six–vertex model

\[
1 \quad 1 \quad b_1 \quad b_2 \quad 1 - b_1 \quad 1 - b_2
\]

Until the quadrant is filled
Stochastic six–vertex model

\[
\begin{array}{cccccc}
1 & 1 & b_1 & b_2 & 1 - b_1 & 1 - b_2 \\
\end{array}
\]

The resulting paths are level lines of the height function

- Height is 0 at the origin,
- increases up,
- decreases to the right.
Theorem. (Borodin–Corwin–Gorin-14) For domain–wall boundary conditions and fixed $0 < b_2 < b_1 < 1$, $\frac{1}{L}H(Lx, Ly) \to h(x, y)$ with fluctuations on $L^{1/3}$ scale given by the Tracy–Widom distribution.

$TW = \text{universal law for the largest eigenvalue of Hermitian matrices and for particle system in Kardar–Parisi–Zhang class}$
Domain–wall and fixed weights

\[ s = \frac{1 - b_1}{1 - b_2} \]

\[
\frac{1}{L} H(Lx, Ly) \rightarrow \eta(x, y) =
\begin{cases}
0, & \frac{x}{y} > s^{-1}, \\
\frac{(\sqrt{s}x - \sqrt{y})^2}{1 - s}, & s \leq \frac{x}{y} \leq s^{-1} \\
y - x, & \frac{x}{y} < s.
\end{cases}
\]

**Theorem.** (Borodin–Corwin–Gorin-14) For domain–wall boundary conditions and fixed \(0 < b_2 < b_1 < 1\), \(\frac{1}{L} H(Lx, Ly) \rightarrow \eta(x, y)\) with fluctuations on \(L^{1/3}\) scale given by the Tracy–Widom distribution.

\[
\rho_x \cdot s + \rho_y \cdot (s + (s - 1)\rho)^2 = 0, \quad \rho = \eta_x.
\]

*1st–order non-linear* PDE (Gwa–Spohn–92) (Reshetikhin–Sridhar–16)
### Domain–wall and fixed weights

\[ s = \frac{1 - b_1}{1 - b_2} \]

\[
\frac{1}{L} H(Lx, Ly) \to \mathfrak{h}(x, y) = \begin{cases} 
0, & \frac{x}{y} > s^{-1}, \\
\frac{(\sqrt{s}x - \sqrt{y})^2}{1 - s}, & s \leq \frac{x}{y} \leq s^{-1} \\
y - x, & \frac{x}{y} < s.
\end{cases}
\]

\[
\rho_x \cdot s + \rho_y \cdot (s + (s - 1)\rho)^2 = 0, \quad \rho = \mathfrak{h}_x
\]

- Instead of \( b_1, b_2 \), only \( s \). Where is the second parameter?
- Where is the linear telegraph equation?
Domain–wall and fixed weights

\[ s = \frac{1 - b_1}{1 - b_2} \]

\[ \frac{1}{L} H(Lx, Ly) \rightarrow h(x, y) = \begin{cases} 
0, & \frac{x}{y} > s^{-1}, \\
\frac{\sqrt{s x} - \sqrt{y}}{1 - s}, & s \leq \frac{x}{y} \leq s^{-1} \\
y - x, & \frac{x}{y} < s.
\end{cases} \]

\[ \rho_x \cdot s + \rho_y \cdot (s + (s - 1)\rho)^2 = 0, \quad \rho = h_x \]

- Instead of \( b_1, b_2 \), only \( s \). Where is the second parameter?
- Where is the linear telegraph equation?

(Borodin–Gorin–18): One needs to rescale weights \( b_1, b_2 \).
**Theorem.** (Borodin–Gorin–18) For arbitrary boundary conditions and rescaled weights, \( \frac{1}{L} H(Lx, Ly) \to h(x, y) \) with

\[
\frac{\partial^2}{\partial x \partial y} \left( q_h(x, y) \right) + \beta_2 \frac{\partial}{\partial x} \left( q_h(x, y) \right) + \beta_1 \frac{\partial}{\partial y} \left( q_h(x, y) \right) = 0,
\]

\[
q_h(x, 0) = \chi(x), \quad q_h(0, y) = \psi(y).
\]
\[
\begin{align*}
\text{Rescaled weights: Law of Large Numbers} \\
\begin{array}{cccccccc}
5 & 3 & \text{ } & 2 \\
4 & 2 \text{ } & 1 \\
3 & 1 \text{ } & 0 \\
2 & \text{ } -1 & \text{ } -2 \\
1 & \text{ } 0 \text{ } & \text{ } -1 \text{ } & \text{ } -2 \\
0 & \text{ } 1 \text{ } & \text{ } 2 \text{ } & \text{ } 3 \text{ } & \text{ } 4 \text{ } & \text{ } 5 \text{ } & \cdots
\end{array}
\end{align*}
\]

\[
b_1 = \exp \left( -\frac{\beta_1}{L} \right)
\]

\[
b_2 = \exp \left( -\frac{\beta_2}{L} \right)
\]

\[
q = \left( \frac{b_2}{b_1} \right)^L = e^{\beta_1 - \beta_2}
\]

\[
\left( q^{\delta(x,y)} \right)_{xy} + \beta_2 \left( q^{\delta(x,y)} \right)_x + \beta_1 \left( q^{\delta(x,y)} \right)_y = 0
\]

\[
q^{\delta(x,0)} = \chi(x), \quad q^{\delta(0,y)} = \psi(y).
\]

- In $t = x + y$, $z = x - y$, a version of the \textbf{Telegraph equation}.
- \textbf{Characteristic} Cauchy problem has a unique solution.
Rescaled weights: Law of Large Numbers

Low density of corners

\[ b_1 = \exp \left( -\frac{\beta_1}{L} \right) \]
\[ b_2 = \exp \left( -\frac{\beta_2}{L} \right) \]
\[ q = \left( \frac{b_2}{b_1} \right)^L = e^{\beta_1 - \beta_2} \]

\[
(q^h(x,y))_{xy} + \beta_2 (q^h(x,y))_x + \beta_1 (q^h(x,y))_y = 0
\]

2nd order hyperbolic limit shape equation is strange:

- Diffusions: parabolic equations (e.g. Brownian motion)
- Interacting particle systems: 1st order equations (e.g. TASEP)
- 2d statistical mechanics: 2nd order elliptic Euler–Lagrange equations through variational principles (e.g. random tilings)
Rescaled weights: Fluctuations

What about fluctuations $H(Lx, Ly) - \mathbb{E}H(Lx, Ly)$?

Reminder. For fixed $b_1, b_2$, they were Tracy–Widom on $L^{1/3}$ scale.
Rescaled weights: Fluctuations

\[ b_1 = \exp \left( -\frac{\beta_1}{L} \right) \quad b_2 = \exp \left( -\frac{\beta_2}{L} \right) \]

\[ q = \frac{b_2}{b_1} = q^{1/L} \]

Claim. \( H(Lx, Ly) - \mathbb{E}H(Lx, Ly) \approx L^{1/2} \times \text{Gaussian} \).
Rescaled weights: Fluctuations

\[ b_1 = \exp \left( -\frac{\beta_1}{L} \right) \quad b_2 = \exp \left( -\frac{\beta_2}{L} \right) \]

\[ q = \frac{b_2}{b_1} = q^{1/L} \]

Claim. \( H(Lx, Ly) - \mathbb{E} H(Lx, Ly) \approx L^{1/2} \times \text{Gaussian} \).

Theorem. (Borodin–Gorin–18, Shen–Tsai–18) \( \lim_{L \to \infty} \frac{1}{L} \left( q^H(Lx,Ly) - \mathbb{E} q^H(Lx,Ly) \right) \) solves \textit{Stochastic Telegraph}

\[ \dot{\phi}_{xy} + \beta_2 \phi_x + \beta_1 \phi_y = \dot{\mathcal{W}} \sqrt{V(x,y)} \]

\[ V(x, y) = (\beta_1 + \beta_2) q^b_x q^b_y + (\beta_2 - \beta_1) \beta_2 q^b_x q^b_y - (\beta_2 - \beta_1) \beta_1 q^b_x q^b_y \]

R.H.S. = \text{2d white noise} \times \text{non-linear functional of the limit shape}
Stochastic six–vertex model in the quadrant in low corner density asymptotic regime.

- Deterministic limit (LLN) for $q^H(x,y)$ is given by the homogeneous Telegraph equation.
- Gaussian fluctuations (CLT) are given by Stochastic Telegraph equation.

Why?
Feynman–Kac for Heat equation

For a second, switch to (parabolic) **Heat equation**.

\[ H_t = \frac{1}{2} H_{xx}, \quad t \geq 0, \quad H(0, x) = f(x). \]

The Feynman–Kac formula expresses the solution:

\[ H(t, x) = \mathbb{E}f(B_t), \]

where \( B_t \) is the **Brownian motion** started at \( B_0 = x \).

Similar representation is possible for the Telegraph equation!
Feynman–Kac for Telegraph

**Persistent random walk.**

Intensity $\beta_1$

Intensity $\beta_2$

Turns down/to the left at Poisson random times.
Theorem. (Borodin–Gorin–18; following Goldstein–51, Kac–74)

\[ \phi_{xy} + \beta_2 \phi_x + \beta_1 \phi_y = u, \quad x, y > 0; \quad \phi(x, 0) = \chi(x), \quad \phi(0, y) = \psi(y). \]

Then random characteristics solve the inhomogeneous Telegraph:

\[ \phi(X, Y) = \mathbb{E}\chi(\hat{x}) + \mathbb{E}\psi(\hat{y}) + \mathbb{E} \left[ \int_0^X \int_0^Y I_{\text{between}}(x, y) u(x, y) \, dx \, dy \right]. \]
Six–vertex and persistent random walks

\[ b_1 = \exp\left(-\frac{\beta_1}{L}\right) \quad b_2 = \exp\left(-\frac{\beta_2}{L}\right) \]

If paths are **rare** (low density limit), then each of them becomes a persistent random walk. They are essentially independent.
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**Conclusion.** LLN/CLT for the height at **low density** is the same as LLN/CLT for a family of independent persistent random walks. Hence, connection to the Telegraph equation.
Six–vertex and persistent random walks

If paths are rare (low density limit), then each of them becomes a persistent random walk. They are essentially independent.

**Conclusion.** LLN/CLT for the height at low density is the same as LLN/CLT for a family of independent persistent random walks. Hence, connection to the Telegraph equation.

How to explain the same connection at high densities and the appearance of $q^{H(x,y)}$?
Summary

- Stochastic six–vertex model in the rare corners regime unexpectedly connects to hyperbolic PDEs.
- \( \lim_{L \to \infty} q^{H(L_x,L_y)} \) solves \( L \to \infty \) Telegraph equation.
- \( q \to 0 \): fixed weights 1st order nonlinear PDE for LLN.
- \( \lim_{L \to \infty} \sqrt{L}(q^{H} - \mathbb{E} q^{H}) \) — Stochastic Telegraph.
- Links to persistent random walks — Feynman–Kac formula for Telegraph.

\[ \phi_{xy} + \beta_2 \phi_x + \beta_1 \phi_y = \dot{W} \sqrt{V} \]
Main tool: four point relation

Proofs rely on **exact discrete analogue** of stochastic Telegraph.

\[
\begin{array}{cccccc}
1 & 1 & b_1 & b_2 & 1 - b_1 & 1 - b_2 \\
\hline
H & H & H + 1 & H & \phantom{H - 1} & \phantom{H - 1} \\
H & H & H & H - 1 & \phantom{H - 1} & \phantom{H - 1} \\
\phantom{H} & \hline
\phantom{H} & \phantom{H + 1} & \phantom{H - 1} & H & \phantom{H - 1} & \phantom{H - 1}
\end{array}
\]

**Theorem.** (Borodin–Gorin–18; with help of Wheeler) For the stochastic six–vertex model in the quadrant with **arbitrary** boundary conditions, and each \(x, y = 1, 2, \ldots\), set for \(q = \frac{b_2}{b_1}\):

\[
\xi(x, y) = q^{H(x,y)} - b_1 q^{H(x-1,y)} - b_2 q^{H(x,y-1)} + (b_1 + b_2 - 1) q^{H(x-1,y-1)}.
\]

Then \(\xi\) is a **martingale** with explicit variance:

1. \(\mathbb{E}[\xi(x, y) \mid H(u, v), u < x \text{ or } v < y] = 0\).
2. \(\mathbb{E}[\xi^2(x, y) \mid H(u, v), u < x \text{ or } v < y] = (b_1(1 - b_1) + b_1(1 - b_2)) \Delta_x \Delta_y + b_1(1 - b_2)(1 - q)q^{H(x,y)} \Delta_x - b_1(1 - b_1)(1 - q)q^{H(x,y)} \Delta_y\),

with \(\Delta_x = q^{H(x,y-1)} - q^{H(x-1,y-1)}\), \(\Delta_y = q^{H(x-1,y)} - q^{H(x-1,y-1)}\).