Singularity Formation in General Relativity

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The Einstein-vacuum equations on $\mathbb{R} \times \mathbb{T}^D$

$$\boxed{\mathbf{Ric}_{\mu\nu} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\mu\nu} = 0}$$

- Data on $\Sigma_1 = \mathbb{T}^D$ are tensors $(\mathring{g}, \mathring{k})$ verifying the Gauss and Codazzi constraints
- The value of D is entertaining; stay tuned
- Our data will be Sobolev-close to Kasner data
- Choquet-Bruhat and Geroch: data verifying constraints launch a unique maximal globally hyperbolic development $(\mathcal{M}, \mathbf{g})$

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Kasner solutions

$$\mathbf{g}_{KAS} = -dt \otimes dt + \sum_{i=1}^{D} t^{2q_i} dx^i \otimes dx^i$$

The $q_i \in (-1, 1]$ verify the Kasner constraints:

$$\sum_{i=1}^{D} q_i = 1, \qquad \sum_{i=1}^{D} (q_i)^2 = 1$$

$$\mathbf{Riem}_{\alpha\beta\gamma\delta}\mathbf{Riem}^{\alpha\beta\gamma\delta} = Ct^{-4}$$

where $C > 0$ (unless a q_i is equal to 1)

“Big Bang” singularity at $t = 0$

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Hawking's incompleteness theorem

Theorem (Hawking (specialized to vacuum))

Assume

- $(\mathcal{M}, \mathbf{g})$ *is the maximal globally hyperbolic development of data* $(\mathring{g}, \mathring{k})$ *on* $\Sigma_1 \simeq \mathbb{T}^D$
- $\mathrm{tr}\mathring{k} < C < 0$

Then no past-directed timelike geodesic emanating from Σ_1 *is longer than* $C' < \infty$.

- Hawking's theorem applies to perturbations of Kasner data

Glaring question: Why are the timelike geodesics incomplete?

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Main theorem

Theorem (JS and I. Rodnianski)

For Sobolev-class perturbations of the data (at $t = 1$) of Kasner solutions with

$$\boxed{\max_{i=1}^{D} |q_i| < \frac{1}{6},}$$

the past-incompleteness is caused by spacetime curvature blowup: $\mathbf{Riem}_{\alpha\beta\gamma\delta}\mathbf{Riem}^{\alpha\beta\gamma\delta} \sim Ct^{-4}$.

- *Such Kasner solutions exist when $D \geq 38$.*

- First stable spacelike singularity formation result in GR without symmetry as an effect of pure gravity.
- Qualitatively, the blowup is very different than the weak null singularities of Dafermos and Luk.
- Previously, we proved related stable spacelike singularity formation results for nearly spatially isotropic (i.e., near-FLRW) solutions to the Einstein-scalar field system with $D = 3$.
- The new techniques can be applied to various Einstein matter systems with $D = 3$ for

moderately spatially anisotropic data

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Other contributors

Many people have investigated solutions to Einstein's equations near spacelike singularities:

Partial list of contributors

Aizawa, Akhoury, Andersson, Anguige, Aninos, Antoniou, Barrow, Béguin, Berger, Beyer, Chitré, Claudel, Coley, Cornish, Chrusciel, Damour, Demaret, Eardley, Ellis, Elskens, van Elst, Garfinkle, Goode, Grubišić, Heinzle, Henneaux, Hsu, Isenberg, Kichenassamy, Koguro, LeBlanc, LeFloch, Levin, Liang, Lim, Misner, Moncrief, Newman, Nicolai, Reiterer, Rendall, Ringström, Röhr, Sachs, Saotome, Spindel, Ståhl, Tod, Trubowitz, Uggla, Wainwright, Weaver, $\cdots$

Einstein's equations in CMCTSC gauge

Decomposing $\mathbf{g} = -n^2 dt \otimes dt + g_{ab} dx^a \otimes dx^b$, Einstein's equations with $k^a_{\ a} = -t^{-1}$ are:

$$\boxed{\begin{aligned} \partial_t g_{ij} &= -2n g_{ia} k^a_{\ j}, \\ \partial_t (k^i_{\ j}) &= -\nabla^i \nabla_j n + n \left(\mathrm{Ric}^i_{\ j} - t^{-1} k^i_{\ j} \right), \\ \Delta_g (n-1) &= t^{-2}(n-1) + n\mathrm{R} \end{aligned}}$$

subject to the constraints

$$\boxed{\begin{aligned} \mathrm{R} - k^a_{\ b} k^b_{\ a} + t^{-2} &= 0, \\ \nabla_a k^a_{\ i} &= 0 \end{aligned}}$$

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Analysis outline

The hard part is showing that the solution exists all the way to $t = 0$. The key is to prove: $|tk^i{}_j(t, x)|$ is bounded.

- Low-norm bootstrap assumptions (slightly worse than Kasner): $\|g_{ij}\|_{L^\infty(\Sigma_t)} \le t^{-1/3}$, $\|g^{ij}\|_{L^\infty(\Sigma_t)} \le t^{-1/3}$
- High-norm bootstrap assumptions: $\|g\|_{\dot{H}^{N+1}(\Sigma_t)} \le \epsilon t^{-A}$, $\|k\|_{\dot{H}^{N}(\Sigma_t)} \le \epsilon t^{-A}$
- N and A are parameters, with A large and N chosen large relative to A
- ϵ chosen small relative to N and A
- Interpolation: $\|\partial_i g_{jk}\|_{L^\infty(\Sigma_t)} \lesssim \epsilon t^{-(1/3+\delta)}$, where $\delta = \delta(N, A) \to 0$ as $N \to \infty$ with A fixed
- $\partial_t(tk^i{}_j) = t\mathrm{Ric}^i{}_j + \cdots \sim tg^{-3}(\partial g)^2 + \cdots \lesssim \epsilon t^{-(2/3+2\delta)}$
- Thus, integrability of $t^{-(2/3+2\delta)}$ implies that for $t \in (0, 1]$: $|tk^i{}_j(t, x) - k^i{}_j(1, x)| \lesssim \epsilon$

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Top-order energy estimates

For $t \in (0, 1]$, we have:

$$
\begin{aligned}
& \|t^{A+1} g\|^2_{\dot{H}^{N+1}(\Sigma_t)} + \|t^A k\|^2_{\dot{H}^N(\Sigma_t)} \\
& \leq \text{Data} \\
& + \{C_\star - 2A\} \int_t^1 s^{-1} \left\{ \|s^{A+1} g\|^2_{\dot{H}^{N+1}(\Sigma_s)} + \|s^A k\|^2_{\dot{H}^N(\Sigma_s)} \right\} ds \\
& + \cdots ,
\end{aligned}
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where

- $C_\star$ can be large but is independent of N and A
- $\cdots$ denotes lower-order or time-integrable error terms
- In my earlier work with Rodnianski, we had $C_\star = \mathcal{O}(\epsilon)$; "approximate monotonicity"

For A large, the integral has a friction sign

- Hence, can show $\|t^{A+1} g\|^2_{\dot{H}^{N+1}(\Sigma_t)} + \|t^A k\|^2_{\dot{H}^N(\Sigma_t)} \leq \text{Data}$
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Future directions

- Lowering the value of D: heuristics suggest that similar results might hold for $D \geq 10$
- What happens when there is severe spatial anisotropy?
- In particular, are there stable spacelike Einstein-vacuum singularities when $D = 3$?
- What happens when there is matter with timelike characteristics?