Duality for integrable systems associated to quantum toroidal algebras

Evgeny Mukhin

Indiana University Purdue University Indianapolis

XIX International Congress on Mathematical Physics
Montreal, July 2018
The transfer matrices

Let $U_q$ be your favorite quantum group. Let $R \in U_q \tilde{\otimes} U_q$ be the $R$-matrix satisfying the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$ 

Let $Q \in U_q$ be the twist operator:

$$R(Q \otimes Q) = (Q \otimes Q)R.$$ 

Let $V$ be an admissible $U_q$-module. Then the trace

$$T_V = (\text{Tr}_V \otimes 1) ((Q \otimes 1)R) \in \tilde{U}_q$$

is called the transfer matrix.

**Lemma.** For any admissible modules $V_1, V_2$, the transfer matrices commute:

$$T_{V_1}T_{V_2} = T_{V_2}T_{V_1}.$$
The XXZ type models

Recall: \[ T_V = (\text{Tr}_V \otimes 1)((Q \otimes 1)\mathcal{R}). \]

Thus, the \( R \) matrix gives an embedding of the Grothendieck ring of admissible representations to the quantum group:

\[ T : K_0(\text{Rep } U_q) \to \tilde{U}_q, \quad V \mapsto T_V. \]

The image \( \mathcal{B}_q = \text{Im}(T) \) is the commutative algebra of quantum Hamiltonians. The algebra \( \mathcal{B}_q \) acts on an appropriate class of representations of \( U_q \).

**Problem.** (XXZ type models) Understand the spectrum of \( \mathcal{B}_q \).
The Gaudin type models

The limit $q \to 1$ gives an algebra of quantum Hamiltonians in the corresponding universal enveloping algebra:

$$\mathcal{B} = \lim_{q \to 1} \mathcal{B}_q \in \tilde{U}.$$ 

The limit is not easy. There are alternative constructions (of the same algebra) for affine Lie algebras:

- from the center on the critical level [FFR];
- from Segal-Sugawara vectors in the vacuum modules [M];
- shift of argument method [R].

The algebra $\mathcal{B}$ acts on an appropriate class of representations of $U$.

**Problem.** (Gaudin type models) Understand the spectrum of $\mathcal{B}$. 
The Gaudin type models

The limit $q \to 1$ gives an algebra of quantum Hamiltonians in the corresponding universal enveloping algebra:

$$B = \lim_{q \to 1} B_q \in \tilde{U}.$$ 

The limit is not easy. There are alternative constructions (of the same algebra) for affine Lie algebras:

- from the center on the critical level [FFR];
- from Segal-Sugawara vectors [M];
- shift of argument method [R].

The algebra $B$ acts on an appropriate class of representations of $U$.

**Problem.** (Gaudin type models) Understand the spectrum of $B$. 
The algebras of quantum Hamiltonians

The Gaudin type models

The limit $q \to 1$ gives an algebra of quantum Hamiltonians in the corresponding universal enveloping algebra:

$$
\mathcal{B} = \lim_{q \to 1} \mathcal{B}_q \subseteq \tilde{U}.
$$

The limit is not easy. There are alternative constructions (of the same algebra) for affine Lie algebras:

- from the center on the critical level [FFR];
- from Segal-Sugawara vectors in the vacuum modules [M];
- shift of argument method [R].

The algebra $\mathcal{B}$ acts on an appropriate class of representations of $U$.

**Problem.** (Gaudin type models) Understand the spectrum of $\mathcal{B}$. 
The Gaudin type models

The limit $q \to 1$ gives an algebra of quantum Hamiltonians in the corresponding universal enveloping algebra:

$$\mathcal{B} = \lim_{q \to 1} \mathcal{B}_q \subseteq \tilde{U}.$$ 

The limit is not easy. There are alternative constructions (of the same algebra) for affine Lie algebras:

- from the center on the critical level [FFR];
- from Segal-Sugawara vectors in the vacuum mod [M];
- shift of argument method [R].

The algebra $\mathcal{B}$ acts on an appropriate class of representations of $U$.

**Problem.** (Gaudin type models) Understand the spectrum of $\mathcal{B}$. 

The algebras of quantum Hamiltonians

The Gaudin type models

The limit $q \to 1$ gives an algebra of quantum Hamiltonians in the corresponding universal enveloping algebra:

$$\mathcal{B} = \lim_{q \to 1} \mathcal{B}_q \in \tilde{\mathcal{U}}.$$ 

The limit is not easy. There are alternative constructions (of the same algebra) for affine Lie algebras:

- from the center on the critical level [FFR];
- from Segal-Sugawara vectors in the vacuum modules [M];
- shift of argument method [R].

The algebra $\mathcal{B}$ acts on an appropriate class of representations of $\mathcal{U}$.

Problem. (Gaudin type models) Understand the spectrum of $\mathcal{B}$.
The Gaudin type models

The limit $q \to 1$ gives an algebra of quantum Hamiltonians in the corresponding universal enveloping algebra:

$$\mathcal{B} = \lim_{q \to 1} \mathcal{B}_q \in \tilde{U}.$$ 

The limit is not easy. There are alternative constructions (of the same algebra) for affine Lie algebras:

- from the center on the critical level [FFR];
- from Segal-Sugawara vectors in the vacuum modules [M];
- shift of argument me L. Rybnikov, (06)

The algebra $\mathcal{B}$ acts on an appropriate class of representations of $U$.

**Problem.** (Gaudin type models) Understand the spectrum of $\mathcal{B}$. 

The algebras of quantum Hamiltonians

The Gaudin type models

The limit $q \to 1$ gives an algebra of quantum Hamiltonians in the corresponding universal enveloping algebra:

$$B = \lim_{q \to 1} B_q \in \tilde{U}.$$ 

The limit is not easy. There are alternative constructions (of the same algebra) for affine Lie algebras:

- from the center on the critical level \([FFR]\);
- from Segal-Sugawara vectors in the vacuum modules \([M]\);
- shift of argument method \([R]\).

The algebra $B$ acts on an appropriate class of representations of $U$.

**Problem.** (Gaudin type models) Understand the spectrum of $B$. 
An example

Let $U = \mathfrak{gl}_n[t] = \mathfrak{gl}_n \otimes \mathbb{C}[t]$. We use formal series $e_{ij}(x) = \sum_{s=0}^{\infty} (e_{ij} \otimes t^s) x^{-s-1} \in U[[x^{-1}]]$.

Let $\bar{Q} = \sum_{i=1}^{n} u_i e_{ii}$.

Consider the matrix

$$E_n^u = \begin{pmatrix}
\partial_x - u_1 - e_{11}(x) & -e_{21}(x) & \cdots & -e_{n1}(x) \\
-e_{12}(x) & \partial_x - u_2 - e_{22}(x) & \cdots & -e_{n2}(x) \\
\cdots & \cdots & \cdots & \cdots \\
-e_{1n}(x) & -e_{2n}(x) & \cdots & \partial_x - u_n - e_{nn}(x)
\end{pmatrix}.$$

Expand the row determinant:

$$r \det E_n^u = \partial_x^n + B_1(x) \partial_x^{n-1} + B_2(x) \partial_x^{n-2} + \cdots + B_n(x).$$

Theorem. ([T]) Coefficients of $B_i(x)$ commute and generate the algebra $B_n^u$ of quantum Hamiltonians in $\mathfrak{gl}_n[t]$. 
An example

Let $U = \mathfrak{gl}_n[t] = \mathfrak{gl}_n \otimes \mathbb{C}[t]$.

We use formal series $e_{ij}(x) = \sum_{s=0}^{\infty} (e_{ij} \otimes t^s)x^{-s-1} \in U[[x^{-1}]]$.

Let $\bar{Q} = \sum_{i=1}^{n} u_i e_{ii}$.

Consider the matrix

$$E^u_n = \begin{pmatrix}
\partial_x - u_1 - e_{11}(x) & -e_{21}(x) & \cdots & -e_{n1}(x) \\
-e_{12}(x) & \partial_x - u_2 - e_{22}(x) & \cdots & -e_{n2}(x) \\
\vdots & \vdots & \ddots & \vdots \\
-e_{1n}(x) & -e_{2n}(x) & \cdots & \partial_x - u_n - e_{nn}(x)
\end{pmatrix}.$$

Expand the row determinant:

$$\text{rdet } E^u_n = \partial^n_x + B_1(x) \partial^{n-1}_x + B_2(x) \partial^{n-2}_x + \cdots + B_n(x).$$

Theorem. D. Talalaev, (04) $B_i(x)$ commute and generate the algebra $\mathcal{B}^u_n$ of quantum $\mathfrak{gl}_n$.
An example

Let \( U = \mathfrak{gl}_n[t] = \mathfrak{gl}_n \otimes \mathbb{C}[t] \).

We use formal series \( e_{ij}(x) = \sum_{s=0}^{\infty} (e_{ij} \otimes t^s)x^{-s-1} \in U[[x^{-1}]] \).

Let \( \bar{Q} = \sum_{i=1}^{n} u_i e_{ii} \).

Consider the matrix

\[
E^u_n = \begin{pmatrix}
\partial_x - u_1 - e_{11}(x) & -e_{21}(x) & \ldots & -e_{n1}(x) \\
-e_{12}(x) & \partial_x - u_2 - e_{22}(x) & \ldots & -e_{n2}(x) \\
\ldots & \ldots & \ldots & \ldots \\
-e_{1n}(x) & -e_{2n}(x) & \ldots & \partial_x - u_n - e_{nn}(x)
\end{pmatrix}.
\]

Expand the row determinant:

\[
\text{rdet } E^u_n = \partial_x^n + B_1(x) \partial_x^{n-1} + B_2(x) \partial_x^{n-2} + \cdots + B_n(x).
\]

**Theorem.**([T]) Coefficients of \( B_i(x) \) commute and generate the algebra \( \mathcal{B}^u_n \) of quantum Hamiltonians in \( \mathfrak{gl}_n[t] \).
The classical $\mathfrak{gl}_m - \mathfrak{gl}_n$ duality

Consider the vector space $V = \mathbb{C}[x_{ij}]_{i=1, \ldots, m; j=1, \ldots, n}$. 
The classical $\mathfrak{gl}_m - \mathfrak{gl}_n$ duality

Consider the vector space $V = \mathbb{C}[x_{ij}]_{i=1,\ldots,m \atop j=1,\ldots,n}$.

$$
\begin{pmatrix}
 x_{11} & x_{12} & \cdots & x_{1n} \\
 x_{21} & x_{22} & \cdots & x_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}
$$
The classical $\mathfrak{gl}_m - \mathfrak{gl}_n$ duality

Consider the vector space $V = \mathbb{C}[x_{ij}]_{i=1,\ldots,m}$.

$$
\begin{pmatrix}
x_{11} & x_{12} & \ldots & x_{1n} \\
x_{21} & x_{22} & \ldots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \ldots & x_{mn}
\end{pmatrix}
$$
The classical $\mathfrak{gl}_m - \mathfrak{gl}_n$ duality

Consider the vector space $V = \mathbb{C}[x_{ij}]_{i=1, \ldots, m}$, $j=1, \ldots, n$.

$$
\begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}
$$
The classical $\mathfrak{gl}_m - \mathfrak{gl}_n$ duality

Consider the vector space $V = \mathbb{C}[x_{ij}]_{i=1,\ldots,m}$. 

**Lemma.** As a $\mathfrak{gl}_m$ module, $V = \bigoplus_{k_1,\ldots,k_n=0}^{\infty} L_{k_1 \omega_1}^{(m)} \otimes \cdots \otimes L_{k_n \omega_1}^{(m)}$.

As a $\mathfrak{gl}_n$ module, $V = \bigoplus_{k_1,\ldots,k_m=0}^{\infty} L_{k_1 \omega_1}^{(n)} \otimes \cdots \otimes L_{k_m \omega_1}^{(n)}$.

**Lemma.** We have $[\mathfrak{gl}_m, \mathfrak{gl}_n] = 0$ in $\text{End}(V)$. 

**Lemma.** The collections of matrices $e^{(m)}_{ij}$ and $e^{(n)}_{ij}$ are dual to each other by the identity $\sum_{k=1}^{n} x_{ik} \partial_{jk} = \delta_{ij}$. 

\[ e^{(m)}_{ij} = \sum_{k=1}^{n} x_{ik} \partial_{jk} \quad \text{and} \quad e^{(n)}_{ij} = \sum_{k=1}^{m} x_{ki} \partial_{kj} \]
The $\mathfrak{gl}_n - \mathfrak{gl}_m$ duality of Gaudin models

Choose complex evaluation parameters.

\[
\mathfrak{gl}_m
\]

\[
\mathfrak{gl}_n
\]

\[
e_{ij}^{(m)} = \sum_{k=1}^n x_{ik} \partial_{jk}
\]

\[
e_{ij}^{(n)} = \sum_{k=1}^m x_{ki} \partial_{kj}
\]
The $\mathfrak{gl}_n - \mathfrak{gl}_m$ duality of Gaudin models

Choose complex evaluation parameters.

\[
\begin{pmatrix}
x_{11} & x_{12} & \ldots & x_{1n} \\
x_{21} & x_{22} & \ldots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \ldots & x_{mn}
\end{pmatrix}
\]

\[e_{ij}^{(m)} = \sum_{k=1}^{n} x_{ik} \partial_{jk}\]

\[e_{ij}^{(n)} = \sum_{k=1}^{m} x_{ki} \partial_{kj}\]
The $\mathfrak{gl}_n - \mathfrak{gl}_m$ duality of Gaudin models

Choose complex evaluation parameters.

$$e^{(m)}_{ij}(x) = \sum_{k=1}^{n} \frac{x_{ik} \partial_{jk}}{x - z_k}$$

$$e^{(n)}_{ij}(x) = \sum_{k=1}^{m} \frac{x_{ki} \partial_{kj}}{x - u_k}$$
The $\mathfrak{gl}_n - \mathfrak{gl}_m$ duality of Gaudin models

Choose complex evaluation parameters.

$$
\begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}
$$

$$
\begin{pmatrix}
z_1 \\
z_2 \\
\vdots \\
z_n
\end{pmatrix}
$$

$$
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_m
\end{pmatrix}
$$

Lemma. As a $\mathfrak{gl}_m[t]$ module, $V = \bigoplus_{k_1, \ldots, k_n = 0}^{\infty} L_{k_1 \omega_1}(z_1) \otimes \cdots \otimes L_{k_n \omega_1}(z_n)$.

As a $\mathfrak{gl}_n[t]$ module, $V = \bigoplus_{k_1, \ldots, k_m = 0}^{\infty} L_{k_1 \omega_1}(u_1) \otimes \cdots \otimes L_{k_m \omega_1}(u_m)$. 
The $\mathfrak{gl}_n - \mathfrak{gl}_m$ duality of Gaudin models

Choose complex evaluation parameters.

$$e_{ij}^{(m)}(x) = \sum_{k=1}^{n} \frac{x_{ik} \partial_{jk}}{x-z_k}$$

$$e_{ij}^{(n)}(x) = \sum_{k=1}^{m} \frac{x_{ki} \partial_{kj}}{x-u_k}$$

**Lemma.** As a $\mathfrak{gl}_m[t]$ module, $V = \bigoplus_{k_1,\ldots,k_n=0}^{\infty} L_{k_1\omega_1}(z_1) \otimes \cdots \otimes L_{k_n\omega_1}(z_n).$

As a $\mathfrak{gl}_n[t]$ module, $V = \bigoplus_{k_1,\ldots,k_m=0}^{\infty} L_{k_1\omega_1}(u_1) \otimes \cdots \otimes L_{k_m\omega_1}(u_m).$

**Theorem.** ([MTV]) The algebras of quantum Hamiltonians in $\text{End}(V)$ coincide: $B^u_m = B^z_n.$
The $\mathfrak{gl}_n - \mathfrak{gl}_m$ duality of Gaudin models

Choose complex evaluation parameters.

$$e^{(m)}_{ij}(x) = \sum_{k=1}^{n} \frac{x_{ik} \partial_{jk}}{x - z_k}$$

$$e^{(n)}_{ij}(x) = \sum_{k=1}^{m} \frac{x_{ki} \partial_{kj}}{x - u_k}$$

Lemma. As a $\mathfrak{gl}_m[t]$ module, $V = \bigoplus_{k_1,\ldots,k_n=0}^{\infty} L^{(m)}_{k_1 \omega_1}(z_1) \otimes \cdots \otimes L^{(m)}_{k_n \omega_1}(z_n)$.  

As a $\mathfrak{gl}_n[t]$ module, $V = \bigoplus_{k_1,\ldots,k_m=0}^{\infty} L^{(n)}_{k_1 \omega_1}(u_1) \otimes \cdots \otimes L^{(n)}_{k_m \omega_1}(u_m)$.  

E.M., V. Tarasov, and A. Varchenko, (06)

Theorem. The algebras of quantum Hamiltonians in $\text{End}(V)$ coincide:

$$B^u_m = B^z_n.$$
The $\mathfrak{gl}_n - \mathfrak{gl}_m$ duality of Gaudin models

Choose complex evaluation parameters.

**Lemma.** As a $\mathfrak{gl}_m[t]$ module, $V = \bigoplus_{k_1,\ldots,k_n=0}^{\infty} L^{(m)}_{k_1\omega_1}(z_1) \otimes \cdots \otimes L^{(m)}_{k_n\omega_1}(z_n)$.

As a $\mathfrak{gl}_n[t]$ module, $V = \bigoplus_{k_1,\ldots,k_m=0}^{\infty} L^{(n)}_{k_1\omega_1}(u_1) \otimes \cdots \otimes L^{(n)}_{k_m\omega_1}(u_m)$.

**Theorem.** ([MTV]) The algebras of quantum Hamiltonians in $\text{End}(V)$ coincide: $\mathcal{B}_m^u = \mathcal{B}_n^z$. 
The correspondence of quantum Hamiltonians

Corollary. Eigenvectors of $B^u_m$ and of $B^z_n$ coincide.
The correspondence of quantum Hamiltonians

Corollary. Eigenvectors of $B^u$ and of $B^z$ coincide.

Write: $\prod_{i=1}^{n} (x - z_i) \text{rdet } E^u_m = \sum_{i=1}^{n} \sum_{j=1}^{m} A^{(m)}_{ij} x^i \partial^j$, where $A^{(m)}_{ij} \in \text{End}(V)$.

Write: $\prod_{j=1}^{m} (x - u_i) \text{rdet } E^z_n = \sum_{j=1}^{m} \sum_{i=1}^{n} A^{(n)}_{ji} x^j \partial^i$, where $A^{(n)}_{ij} \in \text{End}(V)$. 
The correspondence of quantum Hamiltonians

\[
\begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}
\]

Corollary. Eigenvectors of \(B^u_m\) and of \(B^z_n\) coincide.

Write: \( \prod_{i=1}^{n} (x - z_i) \det E^u_m = \sum_{i=1}^{n} \sum_{j=1}^{m} A^{(m)}_{ij} x^i \partial^j \)

Write: \( \prod_{j=1}^{m} (x - u_j) \det E^z_n = \sum_{j=1}^{m} \sum_{i=1}^{n} A^{(n)}_{ji} x^j \partial^i \)

Theorem. ([MTV]) We have \( A^{(m)}_{ij} = A^{(n)}_{ji} \).
The correspondence of quantum Hamiltonians

Corollary. Eigenvectors of $B^u_m$ and of $B^z_n$ coincide.

Write: \[ \prod_{i=1}^{n} (x - z_i) \det E_m^u = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij}^{(m)} x^i \partial^j, \text{ where } A_{ij}^{(m)} \in \text{End}(V). \]

Write: \[ \prod_{j=1}^{m} (x - u_i) \det E_n^z = \sum_{j=1}^{m} \sum_{i=1}^{n} A_{ij}^{(n)} x^j \partial^i, \text{ where } A_{ij}^{(n)} \in \text{End}(V). \]

Theorem. ([MTV]) We have $A_{ij}^{(m)} = A_{ji}^{(n)}$.

The correspondence of solutions $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ Bethe ansatz equations is described in [MTV1].
The correspondence of quantum Hamiltonians

Corollary. Eigenvectors of $B^u_m$ and of $B^z_n$ coincide.

Write: $\prod_{i=1}^n (x - z_i) \text{rdet } E^u_m = \sum_{i=1}^n \sum_{j=1}^m A^{(m)}_{ij} x^i \partial^j$, where $A^{(m)}_{ij} \in \text{End}(V)$.

Write: $\prod_{j=1}^m (x - u_i) \text{rdet } E^z_n = \sum_{j=1}^m \sum_{i=1}^n A^{(n)}_{ji} x^j \partial^i$, where $A^{(n)}_{ij} \in \text{End}(V)$.

Theorem. ([MTV]) We have $A^{(m)}_{ij} = A^{(n)}_{ji}$.

The correspondence of solutions $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ Bethe ansatz equations is described by E.M., V. Tarasov, and A. Varchenko, (05).
The correspondence of quantum Hamiltonians

\[ e_{ij}^{(m)}(x) = \sum_{k=1}^{n} \frac{x_{ik} \partial_{jk}}{x - z_k} \]

\[ e_{ij}^{(n)}(x) = \sum_{k=1}^{m} \frac{x_{ki} \partial_{kj}}{x - u_k} \]

**Corollary.** Eigenvectors of \( B_m^u \) and of \( B_n^z \) coincide.

Write: \( \prod_{i=1}^{n} (x - z_i) \text{ rdet } E_m^u = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij}^{(m)} x^i \partial^j \), where \( A_{ij}^{(m)} \in \text{End}(V) \).

Write: \( \prod_{j=1}^{m} (x - u_i) \text{ rdet } E_n^z = \sum_{j=1}^{m} \sum_{i=1}^{n} A_{ji}^{(n)} x^j \partial^i \), where \( A_{ij}^{(n)} \in \text{End}(V) \).

**Theorem.** ([MTV]) We have \( A_{ij}^{(m)} = A_{ji}^{(n)} \).

The correspondence of solutions \( \mathfrak{gl}_m \) and \( \mathfrak{gl}_n \) Bethe ansatz equations is described in [MTV1].
Quantum toroidal algebras

Let $\mathcal{E}_m(q_1, q)$ be the quantum toroidal algebra associated to $\mathfrak{gl}_m$, [GKV].

- The algebra $\mathcal{E}_m(q_1, q)$ is an affinization of $U_q \widehat{\mathfrak{gl}}_m$.
- The algebra $\mathcal{E}_m(q_1, q)$ has generators $E_i(z), F_i(z), K_i^\pm(z), i = 0, \ldots, m - 1$, central element $q^c$ and degree operator $q^d$.
- For any $j$, $E_i(z), F_i(z), K_i^\pm(z) (i \neq j), K_j^\pm(z), q^c, q^d$, generate a subalgebra canonically isomorphic to $U_q \widehat{\mathfrak{gl}}_m$ in Drinfeld new realization. The one for $j = 0$ is called the vertical subalgebra.
- The zero modes $E_{i,0}, F_{i,0}, K_{i,0}^\pm$ generate a subalgebra canonically isomorphic to level zero $U_q \widehat{\mathfrak{gl}}_m$ in Drinfeld-Jimbo realization. It is called the horizontal subalgebra.

Introduce the twist operator $Q = p_0^d \prod_{i=1}^{m-1} p_i^{-\Lambda_i}$.

Let $\widehat{B}_m^p$ be the corresponding algebra of quantum Hamiltonians.
Let $E_m(q_1, q)$ be the quantum toroidal algebra associated to $\mathfrak{gl}_m$, [GKV].

- The algebra $E_m(q_1, q)$ is an affinization of $U_q$. 
- The algebra $E_m(q_1, q)$ has generators $E_i(z), F_i(z), K^\pm_i(z), i = 0, \ldots, m - 1$, central element $q^c$ and degree operator $q^d$.
- For any $j$, $E_i(z), F_i(z), K^\pm_i(z)$ ($i \neq j$), $K^\pm_j(z), q^c, q^d$, generate a subalgebra canonically isomorphic to $U_q \widehat{\mathfrak{gl}}_m$ in Drinfeld new realization. The one for $j = 0$ is called the vertical subalgebra.
- The zero modes $E_{i,0}, F_{i,0}, K^\pm_{i,0}$ generate a subalgebra canonically isomorphic to level zero $U_q \widehat{\mathfrak{gl}}_m$ in Drinfeld-Jimbo realization. It is called the horizontal subalgebra.

Introduce the twist operator $Q = p_0^d \prod_{i=1}^{m-1} p_i^{-\Lambda_i}$.

Let $\hat{B}_m^p$ be the corresponding algebra of quantum Hamiltonians.
Quantum toroidal algebras

Let $E_m(q_1, q)$ be the quantum toroidal algebra associated to $\mathfrak{gl}_m$, [GKV].

- The algebra $E_m(q_1, q)$ is an affinization of $U_q \widehat{\mathfrak{gl}}_m$.
- The algebra $E_m(q_1, q)$ has generators $E_i(z), F_i(z), K_i^\pm(z), i = 0, \ldots, m - 1$, central element $q^c$ and degree operator $q^d$.
- For any $j$, $E_i(z), F_i(z), K_i^\pm(z) (i \neq j), K_j^\pm(z), q^c, q^d$, generate a subalgebra canonically isomorphic to $U_q \widehat{\mathfrak{gl}}_m$ in Drinfeld new realization. The one for $j = 0$ is called the vertical subalgebra.
- The zero modes $E_{i,0}, F_{i,0}, K_{i,0}^\pm$ generate a subalgebra canonically isomorphic to level zero $U_q \widehat{\mathfrak{gl}}_m$ in Drinfeld-Jimbo realization. It is called the horizontal subalgebra.

Introduce the twist operator $Q = p_0^{-d} \prod_{i=1}^{m-1} p_i^{-\Lambda_i}$.

Let $\hat{B}_m^p$ be the corresponding algebra of quantum Hamiltonians.
The Fock module

The quantum toroidal algebra $E_m(q_1, q)$ has a family of Fock representations $F_i(z, t, k)$.

- The Fock module restricted to vertical $U_q \hat{gl}_m$ is the integrable module of level $c = 1$ with highest weight $\Lambda_i$.
- The degree of the highest vector is $t$.
- The central element $q^{\sum_{i=0}^{m-1} \epsilon_i}$ acts by $q^k$.
- The Fock module has a realization by vertex operators, [S].
- The Fock module has a realization by Macdonald type operators, [FJMM].

The quantum Hamiltonian corresponding to module $F_i(z, t, k)$ is computed explicitly. The coefficient $I_s$ of $z^s$ is given by an $mk$-fold integral.

**Example.** For $m = 1$, $I_1 = \int_{|x|=1} F(x) \prod_{s=0}^{\infty} \bar{K}^+(p^{-s}x) dx / x$

(in the region $|q_1| < 1 < |q_1 q^2|$ and by analytic continuation everywhere else).
The Fock module

The quantum toroidal algebra $\mathcal{E}_m(q_1, q)$ has a family of Fock representations $\mathcal{F}_i(z, t, k)$.

- The Fock module restricted to vertical $U_q\widehat{\mathfrak{gl}}_m$ is the integrable module of level $c = 1$ with highest weight $\Lambda_i$.
- The degree of the highest vector is $t$.
- The central element $q^{\sum_{i=0}^{m-1} \epsilon_i}$ acts by $q^k$.
- The Fock module has a realization by vertex operators, Y. Saito, (98).
- The Fock module has a realization by Macdonald type operators, [FJMM].

The quantum Hamiltonian corresponding to module $\mathcal{F}_i(z, t, k)$ is computed explicitly. The coefficient $I_s$ of $z^s$ is given by an $mk$-fold integral.

**Example.** For $m = 1$, $I_1 = \int_{|x| = 1} F(x) \prod_{s=0}^{\infty} \tilde{K}^+(p^{-s} x) dx / x$ (in the region $|q_1| < 1 < |q_1 q_2|$ and by analytic continuation everywhere else).
The Fock module

The quantum toroidal algebra $\mathcal{E}_m(q_1, q)$ has a family of Fock representations $\mathcal{F}_i(z, t, k)$.

- The Fock module restricted to vertical $U_q \widehat{\mathfrak{gl}}_m$ is the integrable module of level $c = 1$ with highest weight $\Lambda_i$.
- The degree of the highest vector is $t$.
- The central element $q \sum_{i=0}^{m-1} \epsilon_i$ acts by $q^k$.
- The Fock module has a realization by vertex operators, [S].
- The Fock module has a realization by Macdonald type operators, [FJMM].

The quantum Hamiltonian corresponding to module $\mathcal{F}_i(z, t, k)$ is computed explicitly. The coefficient $I_s$ of $z^s$ is given by an $mk$-fold integral.

Example. For $m = 1$, $I_1 = \int_{|x|=1} F(x) \prod_{s=0}^{\infty} \bar{K}^+(p^{-s} x) dx / x$

(in the region $|q_1| < 1 < |q_1 q^2|$ and by analytic continuation everywhere else).
The Fock module

The quantum toroidal algebra $\mathcal{E}_m(q_1, q)$ has a family of Fock representations $\mathcal{F}_i(z, t, k)$.

- The Fock module restricted to vertical $U_q \widehat{\mathfrak{gl}}_m$ is the integrable module of level $c = 1$ with highest weight $\Lambda_i$.
- The degree of the highest vector is $t$.
- The central element $q \sum_{i=0}^{m-1} \epsilon_i$ acts by $q^k$.
- The Fock module has a realization by vertex operators. [IS].
- The Fock module has a realization by Macdonald. [B. Feigin, M. Jimbo, T. Miwa, and E.M. (12)]

The quantum Hamiltonian corresponding to module $\mathcal{F}_i$ is computed explicitly. The coefficient $I_s$ of $z^s$ is given by an $mk$-fold integral.

**Example.** For $m = 1$, $I_1 = \int_{|x|=1} F(x) \prod_{s=0}^{\infty} \bar{K}^+(p^{-s}x) dx / x$

(in the region $|q_1| < 1 < |q_1 q^2|$ and by analytic continuation everywhere else).
The Fock module

The quantum toroidal algebra $\mathcal{E}_m(q_1, q)$ has a family of Fock representations $\mathcal{F}_i(z, t, k)$.

- The Fock module restricted to vertical $U_q \hat{gl}_m$ is the integrable module of level $c = 1$ with highest weight $\Lambda_i$.
- The degree of the highest vector is $t$.
- The central element $q \sum_{i=0}^{m-1} \epsilon_i$ acts by $q^k$.
- The Fock module has a realization by vertex operators, [S].
- The Fock module has a realization by Macdonald type operators, [FJMM].

The quantum Hamiltonian corresponding to module $\mathcal{F}_i(z, t, k)$ is computed explicitly. The coefficient $I_s$ of $z^s$ is given by an $mk$-fold integral.

**Example.** For $m = 1$, $I_1 = \int_{|x|=1} F(x) \prod_{s=0}^{\infty} \bar{K}^+(p^{-s}x) dx / x$

(in the region $|q_1| < 1 < |q_1 q^2|$ and by analytic continuation everywhere else).
The $U_q \hat{gl}_m - U_q \hat{gl}_n$ duality

Let $H_{ij}(x)$ be free bosons: $H_{ij}(x)H_{kl}(y) \sim \delta_{ik}\delta_{jl}/(x-y)^2$. Consider the vector space $V = \mathbb{C}[H_{ij}^+(x)]_{ij=1,...,m} \otimes \mathbb{C}(\mathbb{Z}^{mn})$. 
The $U_q \hat{gl}_m - U_q \hat{gl}_n$ duality

Let $H_{ij}(x)$ be free bosons: $H_{ij}(x)H_{kl}(y) \sim \delta_{ik}\delta_{jl}/(x - y)^2$.

Consider the vector space $V = \mathbb{C}[H_{ij}^+(x)]_{i=1,...,m} \otimes \mathbb{C}(\mathbb{Z}^{mn})$.

Then one can define [JF], [FJM]:

$$
\begin{pmatrix}
H_{11}^+(x) & H_{12}^+(x) & \cdots & H_{1n}^+(x) \\
H_{21}^+(x) & H_{22}^+(x) & \cdots & H_{2n}^+(x) \\
\cdots & \cdots & \cdots & \cdots \\
H_{m1}^+(x) & H_{m2}^+(x) & \cdots & H_{mn}^+(x)
\end{pmatrix}
$$
The quantum affine duality

Let $H_{ij}(x)$ be free bosons: $H_{ij}(x)H_{kl}(y) \sim \delta_{ik}\delta_{jl}/(x-y)^2$. Consider the vector space $V = \mathbb{C}[H^+_{11}(x), H^+_{21}(x), \ldots, H^+_{mn}(x)]_{i=1,\ldots,m} \otimes \mathbb{C}(\mathbb{Z}^{mn})$. Then one can define $[JF], [FJM]$.

I. Frenkel and N. Jing, (88)

B. Feigin, M. Jimbo, and E.M., (18)
The $U_q \widehat{\mathfrak{gl}}_m - U_q \widehat{\mathfrak{gl}}_n$ duality

Let $H_{ij}(x)$ be free bosons: $H_{ij}(x)H_{kl}(y) \sim \delta_{ik}\delta_{jl}/(x - y)^2$.

Consider the vector space $V = \mathbb{C}[H_{ij}^+(x)]_{i=1,\ldots,m} \otimes \mathbb{C}(\mathbb{Z}^{mn})$.

Then one can define [JF], [FJM]:

$$
\sum_{k=1}^n \exp(\ldots)
$$

$$
\sum_{k=1}^m \exp(\ldots)
$$
The \( U_q \widehat{gl}_m \)–\( U_q \widehat{gl}_n \) duality

Let \( H_{ij}(x) \) be free bosons: \( H_{ij}(x)H_{kl}(y) \sim \delta_{ik}\delta_{jl}/(x-y)^2 \).

Consider the vector space \( V = \mathbb{C}[H^+_{ij}(x)]_{i=1,\ldots,m} \otimes \mathbb{C}(\mathbb{Z}^{mn}) \).

Then one can define [JF], [FJM]:

\[
U_q \widehat{gl}_m \quad \sum_{k=1}^n \exp(...) \quad \begin{pmatrix}
H^+_{11}(x) & H^+_{12}(x) & \ldots & H^+_{1n}(x) \\
H^+_{21}(x) & H^+_{22}(x) & \ldots & H^+_{2n}(x) \\
\vdots & \vdots & \ddots & \vdots \\
H^+_{m1}(x) & H^+_{m2}(x) & \ldots & H^+_{mn}(x)
\end{pmatrix}
\]

\[ U_q \widehat{gl}_n \quad \sum_{k=1}^m \exp(...) \]

**Lemma.** As a \( U_q \widehat{gl}_m \) module, \( V = \bigoplus_{k_1,\ldots,k_n=0}^{\infty} L^{(m)}_{\Lambda_{r(k_1)}}(t(k_1), k_1) \otimes \cdots \otimes L^{(m)}_{\Lambda_{r(k_n)}}(t(k_n), k_n) \).

Here \( r(k) = \text{res } k \pmod{m} \), \( m \ell(k) = k - r(k) \), \( 2t(k) = r(k)(l(k) + 1) + (m - r(k))l(k)^2 \), \( L^{(m)}_{\Lambda_{r}}(t, k) \) is the integrable module of level 1 with degree of the highest vector \( t \) and central element \( q^{\sum_{i=0}^{m-1} \epsilon_i} \) acting by \( q^k \).
The $U_q \hat{\mathfrak{gl}}_m - U_q \hat{\mathfrak{gl}}_n$ duality

Let $H_{ij}(x)$ be free bosons: $H_{ij}(x)H_{kl}(y) \sim \delta_{ik}\delta_{jl}/(x - y)^2$. Consider the vector space $V = \mathbb{C}[H_{ij}^+(x)]_{i=1,\ldots,m} \otimes \mathbb{C}(\mathbb{Z}^{mn})$. Then one can define [JF], [FJM]:

$$
\begin{pmatrix}
H_{11}^+(x) & H_{12}^+(x) & \ldots & H_{1n}^+(x) \\
H_{21}^+(x) & H_{22}^+(x) & \ldots & H_{2n}^+(x) \\
\vdots & \vdots & \ddots & \vdots \\
H_{m1}^+(x) & H_{m2}^+(x) & \ldots & H_{mn}^+(x)
\end{pmatrix}
$$

Lemma. As a $U_q \hat{\mathfrak{gl}}_m$ module, $V = \bigoplus_{k_1,\ldots,k_n=0}^{\infty} L_{\Lambda r(k_1)}^{(m)}(t(k_1), k_1) \otimes \cdots \otimes L_{\Lambda r(k_n)}^{(m)}(t(k_n), k_n)$. Here $r(k) = \text{res } k \pmod{m}$, $m \ell(k) = k - r(k)$, $2t(k) = r(k)(\ell(k) + 1) + (m - r(k))\ell(k)^2$, $L_{\Lambda r}^{(m)}(t, k)$ is the integrable module of level 1 with degree of the highest vector $t$ and central element $q^{\sum_{i=0}^{m-1} \epsilon_i}$ acting by $q^k$.

Lemma. ([FJM]) We have $[U_q \hat{\mathfrak{gl}}_m, U_q \hat{\mathfrak{gl}}_n] = 0$ in $\text{End}(V)$. 

I. Frenkel and N. Jing, (88) 
B. Feigin, M. Jimbo, and E. M., (18) 
Evgeny Mukhin (IUPUI) 
Duality for quantum toroidal algebras 
Montreal, July 2018
The duality of integrable systems

Choose evaluation parameters

\[ U_q \widehat{\mathfrak{gl}}_m \rightarrow \begin{pmatrix} H_{11}^+(x) & H_{12}^+(x) & \ldots & H_{1n}^+(x) \\ H_{21}^+(x) & H_{22}^+(x) & \ldots & H_{2n}^+(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{m1}^+(x) & H_{m2}^+(x) & \ldots & H_{mn}^+(x) \end{pmatrix} \rightarrow U_q \widehat{\mathfrak{gl}}_n \]
The duality of integrable systems

Choose evaluation parameters

\[
\begin{pmatrix}
H_{11}^+(x) & H_{12}^+(x) & \cdots & H_{1n}^+(x) \\
H_{21}^+(x) & H_{22}^+(x) & \cdots & H_{2n}^+(x) \\
\vdots & \vdots & \ddots & \vdots \\
H_{m1}^+(x) & H_{m2}^+(x) & \cdots & H_{mn}^+(x)
\end{pmatrix}
\]

\[
\begin{align*}
&= z_1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
The duality of integrable systems

Choose evaluation parameters, choose $q_1, q_1^\vee$.

$$\begin{pmatrix} H_{11}^+(x) & H_{12}^+(x) & \cdots & H_{1n}^+(x) \\ H_{21}^+(x) & H_{22}^+(x) & \cdots & H_{2n}^+(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{m1}^+(x) & H_{m2}^+(x) & \cdots & H_{mn}^+(x) \end{pmatrix}$$

$z_1, z_2, \ldots, z_n$

$u_1, u_2, \ldots, u_m$

$U_q \widehat{\mathfrak{gl}}_m \overset{\sim}{\rightarrow} U_q \widehat{\mathfrak{gl}}_n$
The duality of integrable systems

Choose evaluation parameters, choose $q_1, q_1^\vee$. Then one can define:

$$
\mathcal{E}_m(q_1, q) \rightarrow \left( \begin{array}{cccc}
H_{11}^+(x) & H_{12}^+(x) & \cdots & H_{1n}^+(x) \\
H_{21}^+(x) & H_{22}^+(x) & \cdots & H_{2n}^+(x) \\
\vdots & \vdots & \ddots & \vdots \\
H_{m1}^+(x) & H_{m2}^+(x) & \cdots & H_{mn}^+(x)
\end{array} \right) u_1 u_2 \cdots u_m 
$$

$$
\mathcal{E}_n(q_1^\vee, q) \leftarrow
$$
The duality of integrable systems

Choose evaluation parameters, choose $q_1$, $q_1^\vee$. Then one can define:

$$
\mathcal{E}_m(q_1, q) \xrightarrow{\text{z}_1 \quad \text{z}_2 \quad \ldots \quad \text{z}_n}
\begin{pmatrix}
H^+_{11}(x) & H^+_{12}(x) & \ldots & H^+_{1n}(x) \\
H^+_{21}(x) & H^+_{22}(x) & \ldots & H^+_{2n}(x) \\
\ldots & \ldots & \ldots & \ldots \\
H^+_{m1}(x) & H^+_{m2}(x) & \ldots & H^+_{mn}(x)
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\ldots \\
u_m
\end{pmatrix}
$$

Lemma. As an $\mathcal{E}_m(q_1, q)$ module,

$$V = \bigoplus_{k_1, \ldots, k_n = 0}^{\infty} \mathcal{F}^{(m)}_{r(k_1)}(u_1(k_1), t(k_1), k_1) \otimes \cdots \otimes \mathcal{F}^{(m)}_{r(k_n)}(u_n(k_n), t(k_n), k_n).$$

Here $u(k) = (-1)^m(q_1 q)^{-k-m/2} q u$. 

The duality of integrable systems

Choose evaluation parameters, choose \( q_1, q_1^\vee \). Then one can define:

\[
\begin{pmatrix}
H_{11}^+(x) & H_{12}^+(x) & \cdots & H_{1n}^+(x) \\
H_{21}^+(x) & H_{22}^+(x) & \cdots & H_{2n}^+(x) \\
\vdots & \vdots & \ddots & \vdots \\
H_{m1}^+(x) & H_{m2}^+(x) & \cdots & H_{mn}^+(x)
\end{pmatrix}
\]

\[\mathcal{E}_m(q_1, q) \rightarrow \mathcal{E}_n(q_1^\vee, q)\]

Lemma. As an \( \mathcal{E}_m(q_1, q) \) module,

\[
V = \bigoplus_{k_1, \ldots, k_n = 0}^{\infty} \mathcal{F}_{r(k_1)}^{(m)}(u_1(k_1), t(k_1), k_1) \otimes \cdots \otimes \mathcal{F}_{r(k_n)}^{(m)}(u_n(k_n), t(k_n), k_n).
\]

Here \( u(k) = (-1)^m(q_1 q)^{-k-m/2} q u \).

Theorem. ([FJM]) We have \([\hat{\mathcal{B}}_m^p, \hat{\mathcal{B}}_n^p]^\vee = 0\) in \(\text{End}(V)\) provided

\[
p_i = u_{i+1}/u_i, \quad p_i^\vee = z_{i+1}/z_i, \quad p_0 = (q_1^\vee)^n, \quad p_0^\vee = q_1^m.
\]
Conformal limit

Let \( m = 1, n = 2 \). We have \( \mathcal{E}_1(q_1, q) \) and \( \mathcal{E}_2(q_1^\vee, q) \) acting on a two boson space.

Set \( q = 1 - \epsilon/2 + o(\epsilon) \), and then

\[
\begin{align*}
q_1 &= 1 + (1 - r)\epsilon + o(\epsilon), & z_1/z_2 &= 1 - \kappa\epsilon + o(\epsilon), & p_0 &= e^\tau (1 + o(\epsilon)), \\
q_1^\vee &= e^{-\tau} (1 + \epsilon + o(\epsilon)), & p_0^\vee &= 1 + r\epsilon + o(\epsilon), & p_1^\vee &= 1 - \kappa\epsilon/2 + o(\epsilon).
\end{align*}
\]

The limit \( \epsilon \to 0 \) is called Intermediate Long Wave limit.

Further limit \( \tau \to 0 \) is called conformal limit.

In the conformal limit:

- one of the two bosons commutes with all operators in the theory and can be factored out;
- the current \( F(x) \) of \( \mathcal{E}_1(q_1, q) \) is identified to the Virasoro current \( T(z) \);
- the remaining boson is identified with Virasoro Verma module of central charge \( c = 1 - 6(1 - \beta)^2 / \beta \) and highest weight \( h = (\kappa^2 - 1)(1 - \beta)^2 / (4\beta) \), where \( \beta = (r - 1)/r \).
The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows.

The first non-trivial local integral of motion is \( I_2 = \int : T(x)^2 : dx/x \).

**Theorem.** ([FKSW], [FJM1]) The conformal limit of \( \hat{B}_1^p \) coincides with the algebra of local integrals of motion.

It is known that spectrum of \( \hat{B}_1^p \) is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture of [L]:

**Theorem.** ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

\[
\frac{t_i}{t_i - 1} \frac{t_i - \kappa}{t_i - \kappa - 1} \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \frac{t_i - t_j + r}{t_i - t_j - r} \frac{t_i - t_j - r + 1}{t_i - t_j + r - 1} = -1.
\]

This is double Yangian (XXX type) Bethe ansatz equation associated to \( \mathfrak{gl}_1 \). This description is different from the one suggested in [BLZ].
Spectrum of local integrals of motion

The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, also known as quantum KdV flows. The first non-trivial local integral of motion is

$$I_2 = \int : T(x)^2 : \, dx/x.$$ 

**Theorem.** ([FKSW], [FJM1]) The conformal limit of $\hat{B}_1^p$ coincides with the algebra of local integrals of motion.

It is known that spectrum of $\hat{B}_1^p$ is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture of [L]:

**Theorem.** ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

$$\frac{t_i}{t_i - 1} \cdot \frac{t_i - \kappa}{t_i - \kappa - 1} \cdot \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \cdot \frac{t_i - t_j + r}{t_i - t_j - r} \cdot \frac{t_i - t_j - r + 1}{t_i - t_j + r - 1} = -1.$$ 

This is double Yangian (XXX type) Bethe ansatz equation associated to $gl_1$. This description is different from the one suggested in [BLZ].
Spectrum of local integrals of motion

The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows.

The first non-trivial local integral of motion is

\[ I_2 = \int :T(x)^2 : dx/x. \]

**Theorem.** ([FKSW], [FJM1]) The conformal limit of \( \hat{\mathcal{B}}_1^{p} \) coincides with the algebra of local integrals of motion.

It is known that spectrum of \( \hat{\mathcal{B}}_1^{p} \) is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture of [L]:

**Theorem.** ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

\[
\frac{t_i}{t_i - 1} \frac{t_i - \kappa}{t_i - \kappa - 1} \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \frac{t_i - t_j + r}{t_i - t_j - r} \frac{t_i - t_j - r + 1}{t_i - t_j + r - 1} = -1.
\]

This is double Yangian (XXX type) Bethe ansatz equation associated to \( \mathfrak{gl}_1 \). This description is different from the one suggested in [BLZ].
Spectrum of local integrals of motion

The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows.

The first non-trivial local integral of motion is

\[ I_2 = \int : T(x)^2 : \frac{dx}{x}. \]

Theorem. ([FKSW], [FJM1]) The conformal limit of \( \hat{B}_1^p \) coincides with the algebra of local integrals of motion.

It is known that spectrum of \( \hat{B}_1^p \) is given by Bethe ansatz [FMJMM1], [FJMM2]. This gives the conjecture of [L]:

**Theorem.** ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

\[
\frac{t_i}{t_i - 1} \frac{t_i - \kappa}{t_i - \kappa - 1} \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \frac{t_i - t_j + r}{t_i - t_j - r} \frac{t_i - t_j - r + 1}{t_i - t_j + r - 1} = -1.
\]

This is double Yangian (XXX type) Bethe ansatz equation associated to \( \mathfrak{gl}_1 \). This description is different from the one suggested in [BLZ].
Spectrum of local integrals of motion

The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows.

The first non-trivial local integral of motion is \( I_2 = \int :T(x)^2 :dx/x. \)

**Theorem.** ([FKSW], [FJM1]) The conformal limit of \( \hat{\mathcal{B}}_1^p \) coincides with the algebra of local integrals of motion.

It is known that spectrum of \( \hat{\mathcal{B}}_1^p \) is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture of [L]:

**Theorem.** ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

\[
\frac{t_i}{t_i - 1} \frac{t_i - \kappa}{t_i - \kappa - 1} \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \frac{t_i - t_j + r}{t_i - t_j - r} \frac{t_i - t_j - r + 1}{t_i - t_j + r - 1} = -1.
\]

This is double Yangian (XXX type) Bethe ansatz equation associated to \( \mathfrak{gl}_1 \). This description is different from the one suggested in [BLZ].
The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows.

The first non-trivial local integral of motion is \( I_2 = \int : T(x)^2 : \frac{dx}{x} \).

**Theorem.** ([FKSW], [FJM1]) The conformal limit of \( \hat{B}_1^p \) coincides with the algebra of local integrals of motion.

It is known that spectrum of \( \hat{B}_1^p \) is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture of [L]:

**Theorem.** ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

\[
\frac{t_i}{t_i - 1} \frac{t_i - \kappa}{t_i - \kappa - 1} \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \frac{t_i - t_j + r}{t_i - t_j - r} \frac{t_i - t_j - r + 1}{t_i - t_j + r - 1} = -1.
\]

This is double Yangian (XXX type) Bethe ansatz equation associated to \( \mathfrak{gl}_1 \). This description is different from the one suggested in [BLZ].
Spectrum of local integrals of motion

The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows.

The first non-trivial local integral of motion is \( I_2 = \int : T(x)^2 : \, dx/x \).

**Theorem.** ([FKSW], [FJM1]) The conformal limit of \( \hat{B}_1^p \) coincides with the algebra of local integrals of motion.

It is known that spectrum of \( \hat{B}_1^p \) is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture of [L]:

**Theorem.** ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

\[
\frac{t_i}{t_i - 1} \frac{t_i - \kappa}{t_i - \kappa - 1} \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \frac{t_i - t_j + r}{t_i - t_j - r} \frac{t_i - t_j - r + 1}{t_i - t_j + r - 1} = -1.
\]

This is double Yangian (XXX type) Bethe ansatz equation associated to \( \mathfrak{gl}_1 \).

This description is different from the one suggested in [BLZ].
Spectrum of local integrals of motion

The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows.

The first non-trivial local integral of motion is \( I_2 = \int : T(x)^2 : dx/x \).

**Theorem.** ([FKSW], [FJM1]) The conformal limit of \( \hat{\mathcal{B}}_1^p \) coincides with the algebra of local integrals of motion.

It is known that spectrum of \( \hat{\mathcal{B}}_1^p \) is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture A. Litvinov (13)

**Theorem.** ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

\[
\frac{t_i}{t_i - 1} \frac{t_i - \kappa}{t_i - \kappa - 1} \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \frac{t_i - t_j + r}{t_i - t_j - r} \frac{t_i - t_j - r + 1}{t_i - t_j + r - 1} = -1.
\]

This is double Yangian (XXX type) Bethe ansatz equation associated to \( \mathfrak{gl}_1 \). This description is different from the one suggested in [BLZ].
Spectrum of local integrals of motion

The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows.

The first non-trivial local integral of motion is

$$I_2 = \int : T(x)^2 : \frac{dx}{x}.$$

**Theorem.** ([FKSW], [FJM1]) *The conformal limit of $\hat{\mathcal{B}}_1^p$ coincides with the algebra of local integrals of motion.*

It is known that spectrum of $\hat{\mathcal{B}}_1^p$ is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture of [L]:

**Theorem.** ([FJM1]) *The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

$$\frac{t_i}{t_i - 1} \frac{t_i - \kappa}{t_i - \kappa - 1} \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \frac{t_i - t_j + r}{t_i - t_j - r} \frac{t_i - t_j - r + 1}{t_i - t_j + r - 1} = -1.$$

This is double Yangian (XXX type) Bethe ansatz equation associated to $\mathfrak{gl}_1$. This description is different from the one suggested in [BLZ].
Spectrum of local integrals of motion

The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows. The first non-trivial local integral of motion is

$$I_2 = \int : T(x)^2 : \frac{dx}{x}.$$ 

**Theorem.** ([FKSW], [FJM1]) The conformal limit of $\hat{B}_1^p$ coincides with the algebra of local integrals of motion.

It is known that spectrum of $\hat{B}_1^p$ is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture of [L]:

**Theorem.** ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

$$\frac{t_i}{t_i - 1} \frac{t_i - \kappa}{t_i - \kappa - 1} \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \frac{t_i - t_j + r}{t_i - t_j - r} = -1.$$ 

This is double Yangian (XXX type) Bethe ansatz. This description is different from the one suggested in [BLZ].
Spectrum of local integrals of motion

The Virasoro algebra has an algebra of quantum Hamiltonians called local integrals of motion, [FF] also known as quantum KdV flows.

The first non-trivial local integral of motion is $I_2 = \int : T(x)^2 : dx/x$.

**Theorem.** ([FKSW], [FJM1]) The conformal limit of $\hat{B}_1^p$ coincides with the algebra of local integrals of motion.

It is known that spectrum of $\hat{B}_1^p$ is given by Bethe ansatz [FJMM1], [FJMM2]. This gives the conjecture of [L]:

**Theorem.** ([FJM1]) The spectrum of local integral of motion is described by the solutions of Bethe ansatz equation:

$$\frac{t_i}{t_i - 1} \frac{t_i - \kappa}{t_i - \kappa - 1} \prod_{j=1}^{N} \frac{t_i - t_j - 1}{t_i - t_j + 1} \frac{t_i - t_j + r}{t_i - t_j - r} \frac{t_i - t_j - r + 1}{t_i - t_j + r - 1} = -1.$$  

This is double Yangian (XXX type) Bethe ansatz equation associated to $\mathfrak{gl}_1$. This description is different from the one suggested in [BLZ].
One can also define non-local integrals of motion, [BLZ1]. The non-local integrals of motion are given by integrals of products of vertex operators.

**Conjecture.** ([FKSW], [FJM1]) The conformal limit of $\hat{\mathcal{B}}_2^p$ coincides with the algebra of non-local integrals of motion.

The spectrum of $\hat{\mathcal{B}}_2^p$ is also given by Bethe ansatz.

**Conjecture.** ([FJM2]) The spectrum of non-local integrals of motion is described by the solutions of Bethe ansatz equation:

\[
\frac{1}{s_i - 1} + \frac{r - \kappa - 2}{s_i} - \sum_{k=1, k\neq i}^{N} \frac{2}{s_i - s_k} + \sum_{k=1}^{N} \frac{2}{s_i - t_k} = 0, \\
\frac{\kappa - 1}{t_j} - \sum_{k=1, k\neq j}^{N} \frac{2}{t_j - t_k} + \sum_{k=1}^{N} \frac{2}{t_j - s_k} = 0.
\]

These Bethe ansatz equation are Gaudin Bethe ansatz equation associated to affine $\mathfrak{sl}_2$. 
Spectrum of non-local integrals of motion

One can also define non-local integral \( V. \) Bazhanov, S. Lukyanov, ... Integrals of motion are given by integrals and A. Zamolodchikov (96) authors.

**Conjecture.** ([FKSW], [FJM1]) The conformal limit of \( \hat{\mathcal{B}}_2^p \) coincides with the algebra of non-local integrals of motion.

The spectrum of \( \hat{\mathcal{B}}_2^p \) is also given by Bethe ansatz.

**Conjecture.** ([FJM2]) The spectrum of non-local integrals of motion is described by the solutions of Bethe ansatz equation:

\[
\frac{1}{s_i - 1} + \frac{r - \kappa - 2}{s_i} - \sum_{k=1, k \neq i}^{N} \frac{2}{s_i - s_k} + \sum_{k=1}^{N} \frac{2}{s_i - t_k} = 0,
\]

\[
\frac{\kappa - 1}{t_j} - \sum_{k=1, k \neq j}^{N} \frac{2}{t_j - t_k} + \sum_{k=1}^{N} \frac{2}{t_j - s_k} = 0.
\]

These Bethe ansatz equation are Gaudin Bethe ansatz equation associated to affine \( \mathfrak{sl}_2 \).
Spectrum of non-local integrals of motion

One can also define non-local integrals of motion, [BLZ1]. The non-local integrals of motion are given by integrals of products of vertex operators.

**Conjecture.** ([FKSW], [FJM1]) The conformal limit of $\hat{\mathcal{B}}_2^p$ coincides with the algebra of non-local integrals of motion.

The spectrum of $\hat{\mathcal{B}}_2^p$ is also given by Bethe ansatz.

**Conjecture.** ([FJM2]) The spectrum of non-local integrals of motion is described by the solutions of Bethe ansatz equation:

$$\frac{1}{s_i - 1} + \frac{r - \kappa - 2}{s_i} - \sum_{k=1, k\neq i}^N \frac{2}{s_i - s_k} + \sum_{k=1}^N \frac{2}{s_i - t_k} = 0,$$

$$\frac{\kappa - 1}{t_j} - \sum_{k=1, k\neq j}^N \frac{2}{t_j - t_k} + \sum_{k=1}^N \frac{2}{t_j - s_k} = 0.$$

These Bethe ansatz equation are **Gaudin** Bethe ansatz equation associated to affine $\mathfrak{sl}_2$. 
Questions?

Thank you!