short time regularization of diffusive inhomogeneous kinetic equations

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on recent works with R. Alexandre, W.-X. Li, L. Thomann, D. Tonon and I. Tristani

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Introduction

We look at a system described by a density of particles $0 \leq f(t,x,v)$ with $t \geq 0$, $x \in \mathbb{T}^3$ or $\mathbb{R}^3$ and $v \in \mathbb{R}^3$.

Inhomogeneous kinetic equations :

$$\partial_t f + v.\nabla_x f = C(f), \qquad f|_{t=0} = f^0$$

This problem has a long history (Maxwell, Boltzmann, Laudau).

Focus on models when the collision kernel has some diffusion properties

Possible models of diffusive collision kernels $C(f)$ may be

–> Bilinear : Q_B Boltzmann without cutoff, Q_L Landau

–> Linear : L_K Kolmogorov, L_{FP} Fokker-Planck, L_B Boltzmann linéarisé, L_L Landau Linéarisé

For example the historical Kolmogorov equation reads

$$\partial_t f + v\partial_x f = \Delta_v f,$$

–> *hypoellipticity* : Solutions are known to be smooth for positive time

Natural questions

- is it true for others models ?
- what are the applications ?
- are there quantitative estimates ?

Hypoellipticity

Consider the Kolmogorov equation

$$\partial_t f = \Lambda f \qquad \text{with} \qquad \Lambda = -v.\nabla_x + \Delta_v .$$

The theory of (type II) hypoelliptic operators by Hörmander (1967) says that if $U \subset \mathbb{R}^6_{x,v}$ open bounded and $u \in \mathcal{C}_0^\infty(U)$ then

subelliptic estimate

$$\|u\|_s^2 \le C(\|\Lambda u\|_0^2 + \|u\|_0^2) \quad \text{with } s = 2/3$$

Optimal because only $k = 1$ commutator is needed :

$$-\Lambda = X_0 + \sum X_j^* X_j \quad \text{and} \quad \left(X_0, X_j, Y_j \stackrel{\text{def}}{=} [X_j, X_0] \right)$$

span the whole tangent space $T\mathbb{R}^{2n}$ and $s = 2/(2k+1)$.

General remarks about the preceding result :

- A lot of methods exists to get this result (mention Kohn where $s = 1/4$, Hörmander, Helffer-Nourrigat, Rotchild-Stein,....).
- In general local methods.
- $-\Lambda$ not selfadjoint, nor elliptic.

From kinetic considerations we would like :

- Explicit methods and constants.
- Robust methods (apply to other models).
- Look at the time dependent problem $t \longrightarrow S_\Lambda(t)f_0$
- measuring precisely the gain of regularity for the Cauchy problem

First results on the example of the Fokker-Planck equation

$$\partial_t f = \Lambda f \qquad \text{with} \qquad \Lambda = -v.\nabla_x + \nabla_v.(\nabla_v + v) \qquad L_{FP} = \nabla_v.(\nabla_v + v)$$

In three steps :

- global maximal explicit subelliptic estimate (H. Nier 02, Helffer-Nier 05) :

$$\left\| |D_v|^2 u \right\|^2 + \left\| |D_x|^{2/3} u \right\|^2 \lesssim \|\Lambda u\|^2 \lesssim \left\| |D_v|^2 u \right\|^2 + \left\| |D_x|^2 u \right\|^2$$

- Deduce that the spectrum of $-\Lambda \geq 0$ is in $\left\{ |\mathrm{Im}(z)| \lesssim (\mathrm{Re}(z))^3 \right\}$ and get a resolvent estimate outside : cuspidal operators
- Use a Cauchy integral formula

$$S_\Lambda(t) f_0 = \frac{1}{2i\pi} \int_\Gamma e^{-tz} (z + \Lambda)^{-1} f_0 \, dz$$

Using this method

Theorem

$$\textit{for all } r \in \mathbb{R}, \quad \|S_\Lambda(t) f_0\|_{H^{r,r}_{x,v}} \leq \frac{C_r}{t^{N_r}} \|f_0\|_{H^{-r,-r}_{x,v}}$$

- Done for FP in $\mathbb{R}^3$ (H. Nier 02), chains of oscillators (step 2, Eckmann-Hairer 03) general quadratic models (Hitrik, Pravda Starov, Viola 15)...
- Robust proof
- Sometimes sufficient for applications
- But not optimal, decay depends on directions :
  1. Melher Formulas (Green kernels)
  2. Old result concerning Subunit balls, harmonic analysis (Fefferman 83, Coulhon,Saloff-Coste, Varopoulos 92)
  3. Next section.

First examples and Lyapunov functionals

The basic heat equation example

$$\partial_t f - \Delta_v f = 0, \qquad \Lambda = \Delta_v$$

for a density $f(t,v)$ (forget variable x for a moment). Consider a time-dependant functional

$$\mathcal{H}(t,g) = \|g\|^2 + 2t\,\|\nabla_v g\|^2$$

$$\frac{d}{dt}\mathcal{H}(t,f(t)) = -2\,\|\nabla_v f(t)\|^2 + 2\,\|\nabla_v f(t)\|^2 - 2t\,\|\Delta_v f(t)\|^2 \leq 0$$

So that $\|\nabla_v f(t)\|^2 \leq \frac{C_1}{t}\,\|f_0\|^2$ which writes for $\Lambda = \Delta_v$

$$\|S_\Lambda(t) f_0\|_{H^1_v} \leq \frac{C_1}{t^{1/2}}\,\|f_0\|_{L^2_v}$$

We shall do the same for inhomogeneous models using the commutation identity $[\nabla_v, v.\nabla_x] = \nabla_x$.

Consider now the full (conjugated) Fokker Planck equation

$$\partial_t f = \Lambda f \qquad \text{with} \qquad \Lambda = -v.\nabla_x - (-\nabla_v + v).\nabla_v \qquad L_{FP} = -(-\nabla_v + v).\nabla_v$$

For $C > D > E > 1$ to be defined later on, we define the functional

$$\mathcal{H}(t,g) = C\,\|g\|^2 + Dt\,\|\partial_v g\|^2 + Et^2\,\langle \partial_v g, \partial_x g\rangle + t^3\,\|\partial_x g\|^2 .$$

(where the norms are in $L^2(d\mu)$, μ is the Gaussian in velocity).

Then for C, D, E well chosen, we check similarly that

$$\frac{d}{dt}\mathcal{H}(t, f(t)) \leq 0.$$

First note that if $E^2 < D$, the crossed term is controlled by the two others. We have just modified a (time-dependant) norm in H^1.

Some Computations in a simpler case.

▷ First term

$$\frac{d}{dt}\|f\|^2 = 2\langle \partial_t f, f\rangle = -2\langle v\partial_x f, f\rangle - 2\langle (-\partial_v + v)\partial_v f, f\rangle = -2\|\partial_v f\|^2$$

▷ Second term

$$\begin{aligned}\frac{d}{dt}\|\partial_v f\|^2 &= 2\langle \partial_v(\partial_t f), \partial_v f\rangle \\ &= -2\langle \partial_v(v\partial_x f + (-\partial_v + v)\partial_v f), \partial_v f\rangle \\ &= -2\langle v\partial_x\partial_v f, \partial_v f\rangle - 2\langle [\partial_v, v\partial_x] f, \partial_v f\rangle - 2\langle \partial_v(-\partial_v + v)\partial_v f, \partial_v f\rangle . \\ &= -2\langle \partial_x f, \partial_v f\rangle - 2\|(-\partial_v + v)\partial_v f\|^2\end{aligned}$$

▷ Last term

$$\frac{d}{dt}\|\partial_x f\|^2 = -2\|\partial_v\partial_x f\|^2$$

▷ Third important term

$$
\begin{aligned}
&\frac{d}{dt}\langle \partial_x f, \partial_v f\rangle \\
&= -\langle \partial_x (v\partial_x f + (-\partial_v + v)\partial_v f), \partial_v f\rangle - \langle \partial_x f, \partial_v (v\partial_x f + (-\partial_v + v)\partial_v f)\rangle \\
&= -\langle v\partial_x(\partial_x f), \partial_v f\rangle - \langle (-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle \\
&\qquad - \langle \partial_x f, [\partial_v, v\partial_x]\, f\rangle - \langle \partial_x f, v\partial_x\partial_v f\rangle \\
&\qquad - \langle \partial_x f, [\partial_v, (-\partial_v + v)]\,\partial_v f\rangle - \langle (-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle \,.
\end{aligned}
$$

we have

$$\langle v\partial_x\partial_x f, \partial_v f\rangle + \langle \partial_x f, v\partial_x\partial_v f\rangle = 0.$$

and

$$[\partial_v, (-\partial_v + v)] = 1$$

so that

$$\frac{d}{dt}\langle \partial_x f, \partial_v f\rangle = -\|\partial_x f\|^2 + 2\langle (-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle - \langle \partial_x f, \partial_v f\rangle \,.$$

▷ Entropy dissipation inequality (simplest case)

$$\frac{d}{dt}\mathcal{H}(1,f(t)) = -2C\|\partial_v f\|^2 - 2D\|(-\partial_v + v)\partial_v f\|^2 - E\|\partial_x f\|^2 - 2\|\partial_x\partial_v f\|^2 - 2(D+E)\langle\partial_x f, \partial_v f\rangle - 2E\langle(-\partial_v + v)\partial_v f, \partial_x\partial_v f\rangle\,.$$

Therefore, using Cauchy-Schwartz : for $1 < E < D < C$ well chosen,

$$\frac{d}{dt}\mathcal{H}(1,f(t)) \le 0$$

The same occurs with t instead of 1 inside the definition of $\mathcal{H}$. This method, developed first in (H. 05)) gives for any $t \in [0,1)$

Theorem

$$\|S_\Lambda(t)h_0\|_{L^2_x H^1_v} \le \frac{C}{t^{1/2}}\|h_0\|_{L^2_{x,v}}, \qquad \|S_\Lambda(t)h_0\|_{H^1_x L^2_v} \le \frac{C_1}{t^{3/2}}\|h_0\|_{L^2_{x,v}}.$$

The Fractional Kolmogorov case reads

$$\partial_t f = \Lambda f \qquad \text{with} \qquad \Lambda = -v.\nabla_x - (1-\Delta_v)^{s/2} \qquad L_{FK} = -(1-\Delta_v)^{s/2}$$

The same procedure can be applied and we get

Theorem

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$$\|S_\Lambda(t)h_0\|_{L^2_x H^s_v} \le \frac{C}{t^{1/2}}\|h_0\|_{L^2_{x,v}}, \qquad \|S_\Lambda(t)h_0\|_{H^s_x L^2_v} \le \frac{C_1}{t^{(1+2s)/2}}\|h_0\|_{L^2_{x,v}}.$$

The Boltzmann without cutoff case

The Boltzmann equation in the torus reads

$$\partial_t f + v \cdot \nabla_x f = Q_B(f,f)$$

$$\underbrace{(v', v'_*)}_{\text{before collision}} \overset{\longrightarrow}{\longleftarrow} \underbrace{(v, v_*)}_{\text{after collision}}$$

- Conservation of momentum and energy :

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

- Parametrization of (v', v'_*) by an element $\sigma \in S^2$.

$$Q_B(g,f)(v) = \int_{\mathbb{R}^3 \times \mathrm{Spect}^2} \underbrace{B(v - v_*, \sigma)}_{\text{collision kernel}} \big(\underbrace{f(v')\, g(v'_*)}_{\text{"appearing"}} - \underbrace{f(v)\, g(v_*)}_{\text{"disappearing"}} \big)\, dv_*\, d\sigma$$

- Particles interacting according to a repulsive potential of the form $\phi(r) = r^{-(p-1)}$, $p \in (2, +\infty)$. We only deal with the case $p > 5$ (hard potentials).
- The collision kernel $B(v - v_*, \sigma)$ satisfies

$$B(v - v_*, \sigma) = C|v - v_*|^\gamma \, b(\cos\theta), \quad \cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma$$

- b is not integrable on S^2 :

$$\sin\theta \, b(\cos\theta) \approx \theta^{-1-2s}, \quad s = \frac{1}{p-1}, \quad \forall\, \theta \in (0, \pi/2].$$

  For hard potentials $s \in (0, 1/4)$.
- The kinetic factor $|v - v_*|^\gamma$ satisfies $\gamma = \frac{p-5}{p-1}$. For hard potentials $\gamma > 0$.

Near the equilibrium $f = \mu + h$, the Linearized Boltzmann equation reads

$$\partial_t h = \underbrace{-v \cdot \nabla_x h + Q(\mu, h) + Q(h, \mu)}_{\Lambda h = \text{linear part}} \quad (+ \underbrace{Q(h, h)}_{\text{Nonlinear part}}).$$

Theorem (H.-Tonon-Tristani '17)

We have for k large enough and $k' > k$ large enough :

$$\|S_\Lambda(t) h_0\|_{L^2_x H^s_v(\langle v\rangle^k)} \leq \frac{C_s}{t^{1/2}} \|h_0\|_{L^2_{x,v}(\langle v\rangle^{k'})}, \quad \forall\, t \in (0, 1],$$

and

$$\|S_\Lambda(t) h_0\|_{H^s_x L^2_v(\langle v\rangle^k)} \leq \frac{C_r}{t^{(1+2s)/2}} \|h_0\|_{L^2_{x,v}(\langle v\rangle^{k'})}, \quad \forall\, t \in (0, 1].$$

$\hookrightarrow$ Key point to develop our perturbative Cauchy theory.

$\hookrightarrow$ tools In the spirit of [Alexandre-Hérau-Li '15] for the Boltzmann case.

Elements of proof :

- apart from a regularizing part, the linearized Boltzmann Kernel looks like (with $D_v = i^{-1}\nabla_v$)

$$\Lambda \sim -v.\nabla_x + \langle v\rangle^\gamma (1 + |D_v|^2 + |D_v \wedge v|^2 + |v|^2)^s$$

- we can use microlocal/pseudo-differential techniques to estimate the collision part. Anyway, due to bad symbolic properties, Weyl has to be replaced by Wick and Garding inequality by unconditional positivity.
- from Alexandre-Hérau-Li '15, we use symbolic estimates and built a close-to-semiclassical class of symbols.
- a Lyapunov functional very similar to the one of the fractional FP can be built.

The Vlasov-Poisson-Fokker-Planck equation reads

$$\begin{cases} \partial_t f + v.\nabla_x f - (\varepsilon_0 E + \nabla_x V).\nabla_v f - \gamma \nabla_v.\left(\nabla_v + v\right) f = 0, \\ E(t,x) = -\dfrac{1}{|\mathbb{S}^{d-1}|} \dfrac{x}{|x|^d} \star_x \rho(t,x), \quad \text{where} \quad \rho(t,x) = \displaystyle\int f(t,x,v) dv, \\ f(0,x,v) = f_0(x,v), \end{cases}$$

We can write $-\Lambda = v.\nabla_x f - \nabla_x V.\nabla_v f - \gamma \nabla_v.\left(\nabla_v + v\right) f$ and consider the Duhamel formula

$$f(t) = S_\Lambda(t) f_0 + \varepsilon_0 \int_0^t E \underbrace{S_\Lambda(t-s)\nabla_v}_{\text{integrable singularity}} f(s)\, ds.$$

By fixed point Theorem, this yields a result of existence and trend to the equilibrium in $H^{a,a}$ spaces with $a \in (1/2, 2/3)$ (H. Thomann '15)

This type of regularizing result can also be crucial in the Cauchy theory in large spaces as recently proposed by Gualdani-Mischler and Mouhot 15'. We consider here the Boltzmann without cutoff case :

Considering the Boltzmann model, we have

- Conservation of mass, momentum and energy :

$$\int_{\mathbb{R}^3} Q(f,f)(v) \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} dv = 0$$

- Entropy inequality (H-theorem) :

$$D(f) := -\int_{\mathbb{R}^3} Q(f,f)(v) \, \log f(v) \, dv \geq 0$$

and

$$D(f) = 0 \Leftrightarrow f \text{ is a Gaussian in } v)$$

A priori estimates

We fix $\mu = (2\pi)^{-3/2} e^{-|v|^2/2}$.

In what follows, we shall consider initial data f_0 with same mass, momentum, energy as μ

A priori estimates : if f_t is solution of the Boltzmann equation associated to f_0 with finite mass, energy and entropy then :

$$\sup_{t \geq 0} \int \left(1 + |v|^2 + |\log f_t|\right) f_t \, dx \, dv + \int_0^\infty D(f_s) \, ds < \infty.$$

and

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f_t \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} dx \, dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0 \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} dx \, dv.$$

Does $f_t \xrightarrow[t \to \infty]{} \mu$? If yes, what is the rate of convergence ? is it explicit ?

Main results

$$\partial_t f + v \cdot \nabla_x f = Q_B(f,f) \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3.$$

Theorem (Hérau-Tonon-T. '17)

If f_0 is close enough to the equilibrium μ, then there exists a global solution $f \in L^\infty_t(X)$ to the Boltzmann equation. Moreover, for any $0 < \lambda < \lambda_\star$ there exists $C > 0$ such that

$$\forall\, t \geq 0, \quad \|f_t - \mu\|_X \leq C\, e^{-\lambda t}\, \|f_0 - \mu\|_X.$$

- X is a Sobolev space of type $H^3_x L^2_v(\langle v \rangle^k)$ with k large enough.
- $\lambda_\star > 0$ is the optimal rate given by the semigroup decay of the associated linearized operator.
- Key element of the proof in the enlargment theory : Duhamel formula for $\Lambda = A + B$

$$S_\Lambda(t) = S_B(t) + \int_0^t S_\Lambda(t-s) A S_B(s)\, ds.$$

- Global renormalized solutions with a defect measure : DiPerna Lions '89, Villani '96, Alexandre-Villani '04
- Perturbative solutions in $H^{\ell}_{x,v}(\mu^{-1/2})$
  - Landau equation : Guo '02, Mouhot-Neumann '06
  - Boltzmann equation : Gressman-Strain '11, Alexandre et al. '11
- Solutions in Sobolev spaces with polynomial weight for the Boltzmann equation : He-Jiang '17, Alonso et al. '17
- Improvements :
  - The weights are less restrictive.
  - Less assumptions on the derivatives.

Thank you !