Vortex filaments in the 3D Navier-Stokes equations

Jacob Bedrossian
joint work with Pierre Germain and Ben Harrop-Griffiths
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University of Maryland, College Park
Department of Mathematics
Center for Scientific Computation and Mathematical Modeling

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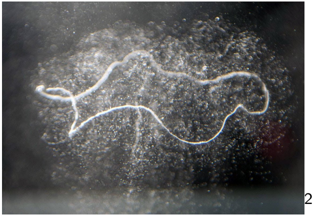
Vortex filaments

- Vortex filaments are one the most common coherent structures in 3D incompressible fluids


[1]

[2]

- Models and analysis for their motion and and behavior have been studied, going back at least to Kelvin in his 1880 work.
- However, the mathematically rigorous derivation of dimension-reduced models, such as the local induction approximation, is not yet developed.

[1] AirTeamImages/Daily Mail UK
[2] Robert Kozloff/University of Chicago

3D Navier-Stokes

- In momentum form

$$\partial_t u + u \cdot \nabla u + \nabla p = \Delta u$$
$$\nabla \cdot u = 0;$$

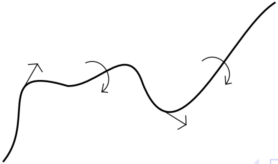
- and in vorticity form for $\omega = \nabla \times u$

$$\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \Delta \omega$$
$$u = \nabla \times (-\Delta)^{-1} \omega.$$

- The scaling symmetry is (hence, L^d is critical for u, $L^{d/2}$ for ω):

$$u(t, y) \mapsto \frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{y}{\lambda}\right), \qquad \omega(t, y) \mapsto \frac{1}{\lambda^2} \omega\left(\frac{t}{\lambda^2}, \frac{y}{\lambda}\right). \tag{1}$$

- Vortex filaments are regions of vorticity highly concentrated along thin tubular neighborhoods:



Mild solutions

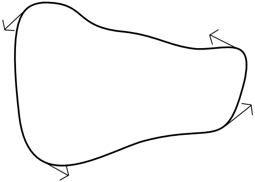
- We will be interested only in *mild solutions* satisfying $\omega \in C^\infty((0,T) \times \mathbb{R}^d)$:

$$\omega(t) = e^{t\Delta}\mu - \int_0^t e^{(t-s)\Delta}\nabla \cdot (u \otimes \omega - \omega \otimes u)ds. \qquad (2)$$

- Generally, well-posedness of mild solutions is closely tied to the scaling symmetry.
- In momentum form, one of largest critical spaces for which one has local well-posedness for *all* data is $u_0 \in L^3$; in vorticity it is $\omega_0 \in L^{3/2}$.

Vortex Filaments as (extra-)critical initial data

- We model vortex filament initial data via measure-valued vorticity directed along a smooth curve γ with constant circulation $\alpha \in \mathbb{R}$.



[3] They also prove something stronger: if the "scaling-critical" piece of the initial data is small, one gets local existence. E.g. if one has a vortex filament with $|\alpha| \ll 1$ and a smooth (but large) background vorticity.

Vortex Filaments as (extra-)critical initial data

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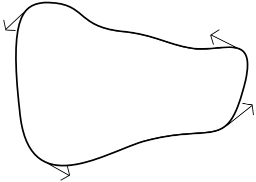

- As observed by Giga-Miyakawa '89, measures of this type are in the scaling-critical Morrey space $\|\mu\|_{M^{3/2}} = \sup_{x,R} R^{-1} |\mu(B(x,R))| < \infty$. They proved global well-posedness for small data in this space[3].
- The associated velocity field is in the Koch-Tataru space BMO^{-1}, but not in L^2_{loc}, so one cannot associate Leray-Hopf weak solutions to this data.
- These two larger critical spaces contain self-similar solutions: *local well-posedness of mild solutions is known only for small data.*

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2D NSE and 3D axisymmetric flows

- The Oseen vortex column:

$$\omega(t, \mathbf{x}, z) = \begin{pmatrix} 0 \\ 0 \\ \frac{\alpha}{4\pi t} e^{-\frac{|\mathbf{x}|^2}{4t}} \end{pmatrix} \qquad (3)$$

  is a self-similar solution to both 2D and 3D Navier-Stokes. In 3D, it is the canonical infinite, straight vortex filament.
- It is known to be unique in the class of 2D measure valued initial data [Gallagher-Gallay-Lions '05, Gallagher/Gallay '05] (in fact the 2D NSE in vorticity form is globally well-posed with measure valued vorticity).
- Gallay-Šverák '15 later considered vortex ring initial data and obtained existence and uniqueness of mild solutions in the axisymmetric class for such initial data (see also Feng/Šverák '15).

Perturbation of the infinite straight filament

Define the space (here $\hat{f}(x,\zeta) = \frac{1}{\sqrt{2\pi}} \int f(x,z) e^{-iz\zeta} dz$),

$$\|f\|_{B_z L^p} = \int \|\hat{f}(\cdot,\zeta)\|_{L^p} d\zeta. \tag{4}$$

Theorem (JB/Germain/Harrop-Griffiths '18)

For all α *and* ω_0 *such that for some* $r \in (1,2)$,

$$\|\omega_0\|_{B_z L^1_x} + \|x \cdot \omega_0^x\|_{B_z L^r \cap B_z L^{\frac{r}{r-1}}} < \infty, \tag{5}$$

there exists a time $T = T(\|\omega_0\|, \alpha)$ *and a mild solution* $\omega \in C_w([0,T); B_z L^1) \cap C^\infty((0,T) \times \mathbb{R}^3)$ *such that*

$$\omega(t,x,z) = \begin{pmatrix} 0 \\ 0 \\ \frac{\alpha}{4\pi t} e^{-\frac{|x|^2}{4t}} \end{pmatrix} + \frac{1}{t}\Omega_c\left(\log t, \frac{x}{\sqrt{t}}, z\right) + \omega_b(t,x,z), \tag{6}$$

satisfying (where $\lim_{T \searrow 0} \epsilon_0 = 0$*),*

$$\sup_{0<t<T} t^{1/4} \|\omega_b(t)\|_{B_z L_x^{4/3}} + \sup_{-\infty<\tau<\log T} \|\langle\xi\rangle^m \Omega_c(\tau)\|_{B_z L^2_\xi} \le \epsilon_0(T). \tag{7}$$

Comments

- Small ω_0 implies global existence ('small' depends on α).
- The proof is a fixed point, so the solutions are automatically unique and stable in the class of solutions whose decomposition admits similar estimates (e.g. filaments with a Gaussian core).
- Rules out the kind of non-uniqueness[4] discussed in Jia/Šverák '13-'14 for self-similar solutions in $L^{3,\infty}$: indeed, the linearization around the filament is stable at *all* α.

[4] Unfortunately, this does *not* imply uniqueness in the general class of mild solutions satisfying suitable a priori estimates. For example, imagine there is a second, fully 3D self-similar solution that looks like e.g. a helical telephone cord twisting at a scale like $O(\sqrt{t})$.

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- The key structure: in self-similar coordinates $\xi = \frac{\mathbf{x}}{\sqrt{t}}$ (note, only in $\mathbf{x}$) the z dependence is almost entirely *subcritical* at the linearized level. This turns the intractable looking 3D stability problem into a perturbation of tractable 2D linearized problems.

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One of the two key linear problems

- The linearization in self-similar variables becomes:

$$\partial_\tau \Omega^\xi + \alpha g \cdot \nabla_\xi \Omega^\xi - \alpha \Omega^\xi \cdot \nabla_\xi g - \alpha e^{\frac{1}{2}\tau} G \partial_z U^\xi = \left(\mathcal{L} + e^\tau \partial_z^2\right) \Omega^\xi$$

$$\partial_\tau \Omega^z + \alpha g \cdot \nabla_\xi \Omega^z + \alpha U^\xi \cdot \nabla_\xi G - \alpha e^{\frac{1}{2}\tau} G \partial_z U^z = \left(\mathcal{L} + e^\tau \partial_z^2\right) \Omega^z,$$

  where $G = e^{-|\xi|^2}$, g is the corresponding velocity, $\mathcal{L} f = \Delta f + \frac{1}{2} \nabla \cdot (\xi f)$.
- After Fourier transforming in z, we can treat this perturbatively as

$$\left(\partial_\tau + e^\tau |\zeta|^2 - \mathcal{L} + \alpha \Gamma\right) w^\xi = \alpha F^\xi$$

$$\left(\partial_\tau + e^\tau |\zeta|^2 - \mathcal{L} + \alpha \Lambda\right) w^z = \alpha F^z,$$

  where

$$\Gamma = g \cdot \nabla_\xi - \nabla_\xi g, \qquad \Lambda = g \cdot \nabla_\xi - \nabla_\xi G \cdot \nabla_\xi^\perp (-\Delta_\xi)^{-1}.$$

- The propagator $e^{t(\mathcal{L} - \alpha \Lambda)}$ was studied by Gallay/Wayne '02 and $e^{t(\mathcal{L} - \alpha \Gamma)}$ by Gallay/Maekawa '11 in their study on 3D stability of the Burgers vortex.

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- The other linear problem we need is the vector transport-diffusion:

$$\partial_t \omega + u_g \cdot \nabla \omega - \omega \cdot \nabla u_g = \Delta \omega, \tag{8}$$

  where $u_g = \frac{1}{\sqrt{t}} g\left(\frac{x}{\sqrt{t}}\right)$.

Decomposition

- Denoting $\omega_g = \frac{1}{4\pi t} e^{-|\mathbf{x}|^2/4t} e_3$.
- We use the decomposition $\omega_c(t, \mathbf{x}, z) = \frac{1}{t}\Omega_c(\log t, \frac{\mathbf{x}}{\sqrt{t}}, z)$,

$$\partial_t \omega_c + \nabla \cdot (u \otimes (\omega_g + \omega_c) - (\omega_g + \omega_c) \otimes u) = \Delta \omega_c \tag{9}$$

$$\omega_c(0) = 0 \tag{10}$$

$$\partial_t \omega_b + \nabla \cdot (u \otimes \omega_b - \omega_b \otimes u) = \Delta \omega_b \tag{11}$$

$$\omega_b(0) = \omega_0. \tag{12}$$

- Then ω_c and ω_b are constructed via fixed point using the two linearizations above to eliminate the linear terms with critical scaling.
- This argument is reminiscent of Gallagher/Gallay '05 and a fixed point variant thereof used in JB/Masmoudi '14.

Perturbation of an arbitrary vortex filament

- Like the z dependence, we expect curvature effects to be subcritical (though that turns out to be hard to make rigorous).
- Let $\gamma : \mathbb{T} \mapsto \mathbb{R}^3$ be a unit-speed parameterization of an arbitrary C^∞, non-self-intersecting closed curve Γ. Define a tubular neighborhood of Γ, Σ_R and the coordinate transform $\Phi : \mathbb{T} \times B(0, R) \to \Sigma_R$.
- Choose an orthonormal frame $(\mathfrak{t}, \mathfrak{n}, \mathfrak{b}) : \mathbb{T} \to \mathbb{R}^3$ along Γ such that $\mathfrak{t} = \gamma'$ and set

$$\Phi(\mathbf{x}, z) = \gamma + x_1\mathfrak{n} + x_2\mathfrak{b}. \tag{13}$$

Perturbation of an arbitrary vortex filament

Theorem (JB/Germain/Harrop-Griffiths '18)

let $\alpha \in \mathbb{R}$, and $\omega_0 \in W^{1,1} \cap W^{1,\infty}$ arbitrary. Then, there is a $T > 0$ and a mild solution $\omega \in C^\infty((0,T) \times \mathbb{R}^3)$ satisfying properties like, for $|x| \leq R/2$:

$$\omega \circ \Phi^{-1} = \begin{pmatrix} 0 \\ 0 \\ \frac{\alpha}{4\pi t} e^{-\frac{|x|^2}{4t}} \end{pmatrix} + \frac{1}{t}\Omega_c\left(\log t, \frac{x}{\sqrt{t}}, z\right) + \omega_b(t,x,z), \qquad (14)$$

where Ω_c and ω_b satisfy similar estimates as in the straight filament case.

- Due to technical difficulties with the anisotropic $B_z L^p$ spaces aligned with the filament, we take ω_0 in a more subcritical space (but not small).
- The uniqueness class we automatically obtain is a little more obscure – we will probably study this a little more before the work appears.

Decomposition

- In the straightened coordinate system $\Delta \mapsto \Delta_\Phi$ has second order error terms of the form $O(|\mathbf{x}|^2)\partial^2$.
- The anisotropic spaces are natural near the filament in the straightened coordinate system, but they don't make sense away from the filament.
- This latter point is an issue because we are taking more regularity in the z direction and less in the $\mathbf{x}$ direction relative to isotropic spaces good for a fixed point (for example $t^{1/4}\|\omega(t)\|_{L^2}$).

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- This latter point is an issue because we are taking more regularity in the z direction and less in the $\mathbf{x}$ direction relative to isotropic spaces good for a fixed point (for example $t^{1/4}\|\omega(t)\|_{L^2}$).
- Split Ω_c and ω_b into ω_{c1}, ω_{c2} and ω_{b1}, ω_{b2}. The ω_{*1} unknowns are constructed in the Σ_R neighborhood in the straightened frame, e.g. $\omega_{c1} = \mathfrak{D}^{-1} J\eta_{c1} \circ \Phi$ for η_{c1} solving a problem similar to Ω_c (hence with Δ instead of the expected Δ_Φ) and then ω_{c2} soaking up the error from Δ in the unstraightened coordinates, using the heat semigroup as the linear propagator.
- All 4 unknowns require a slightly different set of norms.

Thank you for your attention!