ICMP 2018, Montreal

Conformal Field Theory and Critical Phenomena in D=3

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bootstrapcollaboration.com
Critical Phenomena

- CFT
- RG

complementary

- in 2d, since 1980's
- in $d \geq 3$, last ~10 years 'Conformal bootstrap revival'

Review: Poland, Ryckkov, Vichi 1805.04405
Operator Product Expansion (OPE) → Conf. Bootstrap

Mathematically well-defined equations on critical point parameters

(CFT data = critical exponents + more)

Plan:
1. How to derive these eqns
2. How to analyze them
3. Some results
Basic CFT in $d \geq 3$

- CFT = critical point \ lattice artifacts

Example:

3d Ising
$T = T_c$

$S_i = \pm 1$

cubic lattice
\[
\langle s(0) s(r) \rangle \sim \sum_{i=1}^{\infty} \frac{c_i}{r^{\delta_i}} \quad (r \gg a)
\]

+ small rotation-inv breaking

- Junk: \( r \approx a \)
  \( \cdot \) constants \( c_i \)

Universal: \( \delta_i \)
In CFT we introduce 'local operators'

\[
\langle \sigma_i(0) \sigma_i(r) \rangle = \frac{1}{r^{2i}} \quad \gamma_i = 2\Delta_i
\]

at all distances!

One can think of \( \sigma_i \) as linear combinations of lattice operators

\[
S + "S^3" + "S^5" + ... 
\]

But we don't need to know this when using CFT
WHY SO MANY OPERATORS?

**In 2d Ising:** $1, \sigma, \varepsilon$

**In 3d Ising:**
- $1$
- $\sigma_1, \sigma_2, \sigma_3, \ldots$
- $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$
- Tensor operators (non-derivative)
Answer: $\sigma_i$ exist also in 2d but related to $\sigma = \sigma_i$

$\Delta(\sigma_i) - \Delta(\sigma) \in \mathbb{N}$

- In 3d

$\Delta(\sigma) = 0.51814894(10)$

$\Delta(\sigma_2) = 5.2906(11)$

$\eta \xrightarrow{\text{crit. exponent}} \eta = 2 \Delta(\sigma)^{-1}$
Traditionally: Think of $\sigma$ as 'fundamental' and $\sigma_2, \sigma_3, \ldots$ as 'composite' (e.g. $\sigma \sim \varphi$, $\sigma_2 \sim \varphi^5$ in Landau-Ginzburg)

**CFT**: All these operators are equally fundamental (But decoupling – reduced sensitivity to higher-dimension operators)
Any CFT will contain infinitely many operators $\mathcal{O}_i(x)$ characterized by

- $\Delta_i \in \mathbb{R}$ scaling dimension
- $\Pi_i$ $SO(d)$ representation

spectrum of the theory

(no fermions today)
Basics of conf. transformations

Conf. group of $\mathbb{R}^d \cup \{\infty\} = \text{SO}(d+1,1)$

$f : x \mapsto x'$ s.t. $\frac{\partial f^\mu}{\partial x^\nu} = \Omega(x) R^\mu_\nu(x) > 0 \quad \in \text{SO}(d)$
Lie algebra generators

\( P^\mu, M^{\mu \nu}, D, K^\mu \leftrightarrow \) conformal Killing vector fields

e.g.,

\[ \varepsilon (D) = x^\mu \partial_\mu \]

\[ \varepsilon (K^\mu) = 2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu \]
**Conformal group action on**

\[ \mathcal{O} [\Delta, \pi] \]

\[
(pf \circ \mathcal{O})(x') = \Omega(x)^{-\Delta} \pi[R^\mu_\nu(x)] \circ \mathcal{O}(x)
\]

\[
\frac{df^\mu}{dx^\nu} = \Omega(x) R^\mu_\nu(x)
\]

[induced representation \( \rho \) from KDM subgroup preserving \( x=0 \)]

**CONFORMAL INVARiance:**

\[
\langle O_1(x_1) O_2(x_2) ... O_n(x_n) \rangle \in (\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n)^{SO(d+1,1)}
\]
For 2pt functions

\[ \langle O_1(x_1) \, O_2(x_2) \rangle \neq 0 \quad \text{only for} \quad \Delta_1 = \Delta_2 \quad \text{and} \quad \rho_1 = \rho_2^+ \quad \text{(dual reflected)} \]

- Specialize to rank- \( \ell \) tensors
- Assume spectrum nondegenerate

\[ \Rightarrow \quad 0(x) \quad \text{only has 2pt with itself} \]

\[ \langle O^{(\ell)}(x) \, O^{(\ell)}(y) \rangle = \frac{1}{|x-y|^{2\Delta}} \int_{\Delta} I_{\alpha \beta} (x-y) \]

Normalization

unique, known tensor structure for each \( \ell \)
For 3pt functions - finite-dim.

\[ \rho_1 \rightarrow \begin{array}{c} \rho_2 \\ \rho_3 \end{array} = \sum_{i=1}^{I} \lambda_i^\text{CFT} \left( \begin{array}{c} \cdot \\ \Downarrow \text{OPE coeffs.} \end{array} \right)_i \]

CFT data = Spectrum U OPE coeffs
Simple rule to count 3pt functions:

- use conf. transformations to fix 3 points on a line

\[ I = \dim \left( \pi_1 \otimes \pi_2 \otimes \pi_3 \right)^{SO(d-1)} \]

E.g. (scalar - scalar - rank \( \ell \)) gives \( I = 1 + \ell \)
For 4pt functions - $\infty$- dimensional

E.g. for scalars:

$$\langle \sigma_1(x_1) \sigma_2(x_2) \sigma_3(x_3) \sigma_4(x_4) \rangle = K \cdot g(u,v)$$

$K = K(x_i | \Delta_i)$ - fixed

$g(u,v)$ arbitrary

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = u\big|_{1\leftrightarrow 3}$$

conf. inv. cross-ratios
DONE

CONF. INVARIANCE

NEXT

OPERATOR PRODUCT EXPANSION (OPE)

REFLECTION POSITIVITY (UNITARITY)

CONF. BOOTSTRAP
OPE = COMPLETENESS RELATION

\[ \langle \cdot \cdot \cdot_{S^{d-1}} \cdot \cdot \cdot \rangle \]

State \( |\psi\rangle \) in Hilbert space on \( S^{d-1} \)

Full correlation function is scalar product \( \langle \psi_1 | \psi_2 \rangle \)
Local operators at the center generate many states:

Assumption: these states are complete ('state-operator correspondence')
\begin{equation}
\begin{aligned}
\phi_1(x_1) \phi_2(x_2) &= \sum_\sigma \lambda_\sigma \ C^\sigma_{\phi_1 \phi_2}(x_1, x_2) \ \phi_3(x_3) \\
\end{aligned}
\end{equation}

\text{OPE:}
\begin{align*}
\phi_1 \xrightarrow{\lambda_\sigma} \phi_2
\end{align*}

\text{Consistency:}
\begin{align*}
\Rightarrow \text{same } \lambda_\sigma \text{ as in 3pt function}
\end{align*}
OPE must be consistent with conformal invariance.

\[ C_\Omega(x_1, x_2) \] are Clebsch-Gordan coefficients between \( \mathbf{p}_1 \otimes \mathbf{p}_2 \) and representation \( \{ \mathcal{O}(0) \} \equiv \mathbf{p}_0 \).

NB. OPE \( \neq \) tensor product decomposition.
OPE for 4pt functions

\[ \chi = \sum_\sigma \lambda^2_\sigma \]

conformal block

\[ \infty \text{- dim space of invariant 4pt functions} \]
Main Eqn. of Conf. Bootstrap

\[ \sum_{\theta} \lambda^2_{\theta} \left[ \begin{array}{c} \text{Diagram} \end{array} \right] = 0 \]
REFLECTION POSITIVITY

\[ \langle F \Theta(F) \rangle \geq 0 \]

\[ \Rightarrow 2 \text{ types of constraints} \]
- lower bounds on operator dimensions
  In 3d  \[ \Delta (\text{scalars}) \geq \frac{1}{2} \]
  \[ \Delta (\text{rank } l) \geq l+1 \]
- reality constraints on OPE coeffs
  \[ \lambda_\sigma \in \mathbb{R} \]

[kind of obvious in 3d Ising since everything is real]
**CONFORMAL BLOCKS**

\[ K(x_i | \Delta_i) \ G_{\Delta, \ell}(u, v) \]

- Dimension and rank of 'exchanged' \[ \mathcal{O} \]s
- Satisfy 2nd order PDE
  \[ \Rightarrow \] connections to integrability
  [Isachenkov, Schomerus 1602.01858]
- For \( d \) even, explicit \( \frac{\Gamma}{\Gamma} \) solutions
- For \( d \) odd, fast convergent power-series expansion exploiting meromorphic structure (poles at \( \Delta \) outside physical region)
- In practice we compute them numerically with arbitrary precision
**CONVEX GEOMETRY**

Consider bootstrap eqn for 4 identical scalars

\[ \phi \times \phi = 1 + \sum \lambda_\sigma \sigma \]

\[ \sum \frac{\lambda_\sigma^2}{\sigma} \left[ \begin{array}{c} \text{\textbullet} \\ \text{\textbullet} \end{array} \right] = 0 \]

\[ \sum_{\sigma+1} \lambda_\sigma^2 F_\sigma(u,v) = u^{\Delta \phi} - v^{\Delta \phi} \]

For fixed spectrum and varying \( \lambda_\sigma^2 \geq 0 \)

LHS fills a convex cone
Solution exists

\[
\text{RHS}
\]

no solution

\[
\text{RHS}
\]

\[
\text{RHS}
\]

Can prove this by exhibiting a 'separating hyperplane' that is linear functional \( \lambda \) s.t.

\[
\lambda(\text{RHS}) < 0
\]

\[
\lambda(\phi_\theta(u,v)) \geq 0 \quad \forall \theta \in \text{Spec.}
\]
**Convex Geometry for Multiple Correlators**

\[ \langle \phi_i \phi_j \phi_k \phi_l \rangle \quad i, j, k, l = 1 \ldots N \]

Will involve products

\[ \varepsilon_{ij} \varepsilon_{kl} = : M_{AB}^{\varepsilon} \]

\[ A = (ij) \]
\[ B = (kl) \]

**NB:** \( M_{\varepsilon} \succeq 0 \) (positive semi-definite)
$M$ enters linearly in bootstrap eqns:

$$\sum_{\mathbf{0}} \mathcal{L}_0(M) = \text{RHS} \in \mathbb{R}^{N_0} f(u, v)$$

▲ explicit linear operator involving various conf. blocks

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Separating plane conditions:

$\lambda (\text{RHS}) < 0$

$\lambda \circ \mathcal{L}_0 \geq 0 \quad \forall \theta \in \text{Spectrum}$
NUMERICAL STRATEGIES

- Looking for a separating plane is a problem of linear / semidefinite programming

- Efficient algorithms exist (need to modify since $\infty$-dim vectors $f(u,v)$ & $\infty$ many constraints $O_{\Delta,e}$)
CAN CONSTRUCT ORACLES

Spec, OPE assumptions

NO found separating plane

MAYBE

Best current oracle

SDPB [Simmons-Duffin, 1502.02033]
Example:
Is there a unitary 3d CFT with a scalar operator \( \sigma \), \( \Delta_\sigma = \frac{3}{5} \) and OPE \( \sigma \times \sigma = 1 + \lambda_\sigma \epsilon + \ldots \) where the lowest operator \( \epsilon \) has dimension \( \Delta_\epsilon \geq 2 \)

\text{ORACLE: NO!}

(from studying \( \langle \sigma \sigma \sigma \sigma \rangle \))
BOOTSTRAP COMPUTATIONS

- Allowed regions & bounds
- Kinks
- Islands
BOUND: TRY TO MINIMIZE/MAXIMIZE SOMETHING ($\Delta$ or $\lambda$)

Example: $\sigma \times \sigma = 1 + 2\sigma \sigma \epsilon \Delta \sigma + \ldots$

Kink (3d Ising) OPERATOR DECOUPLING
**ISLANDS** require more assumptions (physically motivated)

**Example**
- $\sigma, \epsilon$ only 2 relevant scalars
- $\mathbb{Z}_2$ symmetry

\[
\begin{align*}
\sigma \times \sigma &= 1 + \lambda_{\sigma \sigma} \epsilon + (\Delta \geq 3) \\
\sigma \times \epsilon &= \lambda_{\sigma \epsilon} \sigma + (\Delta \geq 3) \\
\epsilon \times \epsilon &= 1 + \lambda_{\epsilon \epsilon} \epsilon + (\Delta \geq 3)
\end{align*}
\]
study $\langle \sigma \sigma \sigma \sigma \rangle$, $\langle \sigma \sigma \sigma \epsilon \rangle$, $\langle \epsilon \epsilon \epsilon \epsilon \rangle$
Kos, Poland, Simmons-Duffin, Vichi '2016
<table>
<thead>
<tr>
<th>$\mathcal{O}$</th>
<th>$Z_2$</th>
<th>$\ell$</th>
<th>$\Delta$</th>
<th>$f_{\sigma\sigma}$</th>
<th>$f_{\epsilon\epsilon}$</th>
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<tbody>
<tr>
<td>$\epsilon$</td>
<td>+</td>
<td>0</td>
<td>1.412625(10)</td>
<td>1.0518537(41)</td>
<td>1.532435(19)</td>
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<tr>
<td>$\epsilon'$</td>
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<td>0</td>
<td>3.82968(23)</td>
<td>0.053012(55)</td>
<td>1.5360(16)</td>
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<tr>
<td></td>
<td>+</td>
<td>0</td>
<td>6.8956(43)</td>
<td>0.0007338(31)</td>
<td>0.1279(17)</td>
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<tr>
<td></td>
<td>+</td>
<td>0</td>
<td>7.2535(51)</td>
<td>0.000162(12)</td>
<td>0.1874(31)</td>
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<tr>
<td>$T_{\mu\nu}$</td>
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<td>2</td>
<td>3</td>
<td>0.32613776(45)</td>
<td>0.8891471(40)</td>
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<tr>
<td>$T_{\mu'\nu'}$</td>
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<td>2</td>
<td>5.50915(44)</td>
<td>0.0105745(42)</td>
<td>0.69023(49)</td>
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<tr>
<td></td>
<td>+</td>
<td>2</td>
<td>7.0758(58)</td>
<td>0.0004773(62)</td>
<td>0.21882(73)</td>
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<tr>
<td>$C_{\mu\nu\rho\sigma}$</td>
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<td>4</td>
<td>5.022665(28)</td>
<td>0.069076(43)</td>
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<td>+</td>
<td>6</td>
<td>7.028488(16)</td>
<td>0.0157416(41)</td>
<td>0.066136(36)</td>
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<tr>
<th>$\mathcal{O}$</th>
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<th>$\ell$</th>
<th>$\Delta$</th>
<th>$f_{\sigma\sigma}$</th>
<th>-</th>
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<tbody>
<tr>
<td>$\sigma$</td>
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<td>0.5181489(10)</td>
<td>1.0518537(41)</td>
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<td>$\sigma'$</td>
<td>−</td>
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<td>5.2906(11)</td>
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<td>5</td>
<td>6.709778(27)</td>
<td>0.04191549(88)</td>
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</tr>
</tbody>
</table>

Simmons - Duffin
1612.08471

~ 100 operators

What are these numbers?
Exact transcendental critical exponents?

Rong, Su 1807.04939
Atanasov, Hillman, Poland 1807.05702

- Studied $N=1$ SUSY 3d Ising model:

\[
\sigma \times \sigma = 1 + \# \varepsilon + \# \varepsilon'
\]

\[
\varepsilon \times \varepsilon = 1 + \# \sigma + \# \sigma'
\]

Relevant

\[
\Delta \varepsilon = \Delta \sigma + 1
\]

\[
\Delta \varepsilon' = \Delta \sigma' + 1
\]
Found an island:
Conclude with a **bold** conjecture

\[
\frac{\lambda_{\varepsilon\varepsilon\varepsilon}}{\lambda_{\sigma\sigma\varepsilon}} = \tan(1)
\]

\[
\Delta_\sigma = \frac{15 - 2\tan(1) - \sqrt{4\tan(1)^2 + 36\tan(1) + 9}}{18 - 8\tan(1)}
\]

\[
\approx 0.58445133696...
\]
**Conclusions**

- **Conformal bootstrap works**
- **Wealth of precise numerical data about strongly coupled 3D CFTs**

**What's in it for you?**

- Curiosity

- **Try to justify basic assumptions leading to the bootstrap eqns**

- **Analytic understanding of bootstrap bounds, kinks, islands**

↑ That's where help most needed

 Cf. Mazac 1611.10060
 Mazac, Paulos 1803.10233
SPHERE PACKING IN D=8

- Cohn, Elkies:
  numerical upper bounds using linear programming
  \Rightarrow magic function?

- Viazovska, 2016:
  constructed magic function using modular form.

Will this happen for bootstrap?
CONF. INVARIANCE

OPERATOR PRODUCT EXPANSION (OPE)

REFLECTION POSITIVITY (UNITARITY)

CONF. BOOTSTRAP

CONFORMAL BLOCKS

CONVEX GEOMETRY