Recent advances in fluid boundary layer theory

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M∩Φ - ICMP 2018
July 25th, 2018 - Montreal

European Research Council
Established by the European Commission

Outline

Plan

Fluids with small viscosity

Goal: understand the behavior of 2d fluids with small viscosity in a domain $\Omega \subset \mathbf{R}^2$.

$$\begin{aligned} \partial_t \mathbf{u}^\nu + (\mathbf{u}^\nu \cdot \nabla)\mathbf{u}^\nu + \nabla p^\nu - \nu \Delta \mathbf{u}^\nu &= 0 \text{ in } \Omega, \\ \operatorname{div} \mathbf{u}^\nu &= 0 \text{ in } \Omega, \\ \mathbf{u}^\nu_{|\partial\Omega} = 0, \quad \mathbf{u}^\nu_{|t=0} &= \mathbf{u}^\nu_{ini}. \end{aligned} \tag{1}$$

$\rightarrow$ Singular perturbation problem.
Formally, if $\mathbf{u}^\nu \rightarrow \mathbf{u}^E$, and if $\Delta \mathbf{u}^\nu$ remains bounded, then $\mathbf{u}^E$ is a solution of the Euler system

$$\begin{aligned} \partial_t \mathbf{u}^E + (\mathbf{u}^E \cdot \nabla)\mathbf{u}^E + \nabla p^E &= 0 \text{ in } \Omega, \\ \operatorname{div} \mathbf{u}^\nu &= 0 \text{ in } \Omega. \end{aligned} \tag{2}$$

But what about boundary conditions?

Boundary conditions

- **Navier-Stokes:** parabolic system.

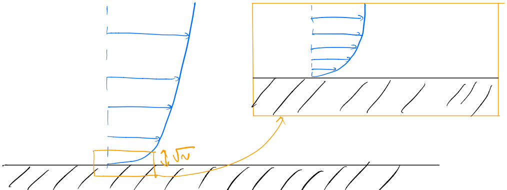
$\rightarrow$ Dirichlet boundary conditions can be enforced: $\mathbf{u}^\nu_{|\partial\Omega} = 0$.

- **Euler:** $\sim$ hyperbolic system, with a divergence-free condition $\operatorname{div} \mathbf{u}^E = 0$.

$\rightarrow$ Condition on the normal component only (non-penetration condition): $\mathbf{u}^E \cdot n_{|\partial\Omega} = 0$.

Consequence:

- Loss of the tangential boundary condition as $\nu \to 0$;
- Formation of a boundary layer in the vicinity of $\partial\Omega$ to correct the mismatch between $0 (= \mathbf{u}^\nu \cdot \tau_{|\partial\Omega})$ and $\mathbf{u}^E \cdot \tau_{|\partial\Omega}$.



The whole space case

Theorem [Constantin& Wu, '96] *If* $\Omega = \mathbf{R}^2$ *or* $\Omega = \mathbb{T}^2$, *any family of Leray-Hopf solutions* $\mathbf{u}^\nu \in \mathcal{C}(\mathbf{R}_+, L^2) \cap L^2(\mathbf{R}_+, H^1)$ *of the Navier-Stokes system converges as* $\nu \to 0$ *towards a solution of the Euler system.*

Proof: energy estimate, by considering $\mathbf{u}^E$ as a solution of Navier-Stokes with a remainder $-\nu\Delta\mathbf{u}^E$.

Consequence: if convergence fails, problems come from the boundary.

The half-space case: Prandtl's Ansatz

Prandtl, 1904: in the limit $\nu \ll 1$, if $\Omega = \mathbf{R}^2_+$,

$$\mathbf{u}^\nu(x,y) \simeq \begin{cases} \mathbf{u}^E(x,y) \text{ for } y \gg \sqrt{\nu} \text{ (sol. of 2d Euler)}, \\ \left(u^P\left(x, \frac{y}{\sqrt{\nu}}\right), \sqrt{\nu} v^P\left(x, \frac{y}{\sqrt{\nu}}\right)\right) \text{ for } y \lesssim \sqrt{\nu}. \end{cases}$$

The velocity field (u^P, v^P) satisfies the Prandtl system

$$\begin{aligned} \partial_t u^P + u^P \partial_x u^P + v^P \partial_Y u^P - \partial_{YY} u^P = -\frac{\partial p^E}{\partial x}(t,x,0) \\ \partial_x u^P + \partial_Y v^P = 0, \\ \mathbf{u}^P_{|Y=0} = 0, \ \lim_{Y\to\infty} u^P(x,Y) = u_\infty(t,x) := u^E(t,x,0), \\ u^P_{|t=0} = u^P_{ini}. \end{aligned}$$

The Prandtl equation: general remarks

$$\begin{aligned} \partial_t u^P + u^P \partial_x u^P + v^P \partial_Y u^P - \partial_{YY} u^P &= -\frac{\partial p^E}{\partial x}(t,x,0) \\ \partial_x u^P + \partial_Y v^P &= 0, \\ \mathbf{u}^P_{|Y=0} = 0, \ \lim_{Y\to\infty} u^P(x,Y) = u_\infty(t,x) &:= u^E(t,x,0), \\ u^P_{|t=0} &= u^P_{ini}. \end{aligned} \tag{P}$$

Comments:

- Nonlocal, scalar equation: write $v^P = -\int_0^Y u^P_x$;
- Pressure is given by Euler flow= data;
- Main source of trouble: nonlocal transport term $v^P \partial_Y u^P$ (loss of one derivative).

Questions around the Prandtl system

1. Is the Prandtl system well-posed? (i.e. does there exist a unique solution?) In which functional spaces? Under which conditions on the initial data?
2. When the Prandtl system is well-posed, can we justify the Prandtl Ansatz? i.e. can we prove that

$$\|\mathbf{u}^\nu - \mathbf{u}^\nu_{\text{app}}\| \to 0 \text{ as } \nu \to 0$$

in some suitable functional space, where the function $\mathbf{u}^\nu_{\text{app}}$ is such that

$$\mathbf{u}^\nu_{\text{app}}(x,y) \simeq \begin{cases} \mathbf{u}^E(x,y) \text{ for } y \gg \sqrt{\nu} \\ \left(u^P\left(x, \frac{y}{\sqrt{\nu}}\right), \sqrt{\nu} v^P\left(x, \frac{y}{\sqrt{\nu}}\right)\right) \text{ for } y \lesssim \sqrt{\nu}. \end{cases}$$

Functional spaces

- **L^2 space:** $\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2\right)^{1/2}$.
- **Sobolev spaces H^s, $s \in \mathbf{N}$:** $\|u\|_{H^s} = \sum_{|k|\leq s} \|\nabla^k u\|_{L^2}$.
- **Space of analytic functions:** $\exists C > 0$, s.t. for all $k \in \mathbf{N}^d$,

$$\sup_{x\in\Omega} |\nabla^k u(x)| \leq C^{|k|+1}|k|!.$$

- **Gevrey spaces G^τ, $\tau > 0$:** $\exists C > 0$, s.t. for all $k \in \mathbf{N}^d$,

$$\sup_{x\in\Omega} |\nabla^k u(x)| \leq C^{|k|+1}(|k|!)^\tau.$$

If $\tau > 1$, G^τ contains non trivial functions with compact support.

Plan

Well-posedness under positivity assumptions

Stationary Prandtl system:

$$\begin{aligned} u\partial_x u + v\partial_Y u - \partial_{YY} u &= -\frac{\partial p^E}{\partial x}(x,0) \\ \partial_x u + \partial_Y v = 0, \quad u_{|x=0} &= u_0 \\ u_{|Y=0} = 0, \quad v_{|Y=0} = 0, \quad \lim_{Y\to\infty} u(x,Y) &= u_\infty(x). \end{aligned} \qquad \text{(SP)}$$

$\sim$ Non-local, "transport-diffusion" equation .

Theorem [Oleinik, 1962]: *Let $u_0 \in \mathcal{C}_b^{2,\alpha}(\mathbf{R}_+)$, $\alpha > 0$. Assume that $u_0(Y) > 0$ for $Y > 0$, $u_0'(0) > 0$, $u_\infty > 0$, and that*

$$-\partial_{YY} u_0 + \frac{\partial p^E}{\partial x}(0,0)) = O(Y^2) \quad \text{for } 0 < Y \ll 1.$$

Then there exists $x^* > 0$ such that (SP) has a unique strong $\mathcal{C}^2$ solution in $\{(x,Y) \in \mathbf{R}^2,\ 0 \le x < x^*,\ 0 \le Y\}$. If $\frac{\partial p^E(x,0)}{\partial x} \le 0$, then $x^* = +\infty$.

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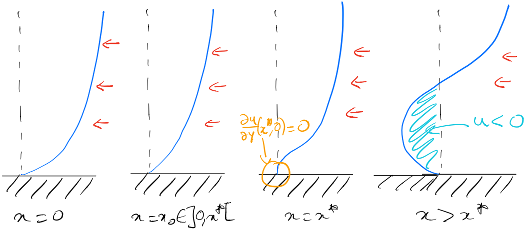
Comments on Oleinik's theorem

- The solution lives as long as there is no recirculation, i.e. as long as u remains positive.
- Proof relies on a nonlinear change of variables [von Mises]: transforms (SP) into a local diffusion equation (porous medium type).
  $\rightarrow$ Maximum principle holds for the new eq. by standard tools and arguments.
- Maximal existence "time" x^*: if $x^* < +\infty$, then
  (i) either $\partial_Y u(x^*, 0) = 0$
  (ii) or $\exists Y^* > 0,\ u(x^*, Y^*) = 0$.
- Monotony (in Y) is preserved by the equation. If u_0 is monotone, scenario (ii) cannot happen.

Illustration of the "separation" phenomenon



Separation point: $\frac{\partial u}{\partial Y}_{|x=x^*,Y=0} = 0.$

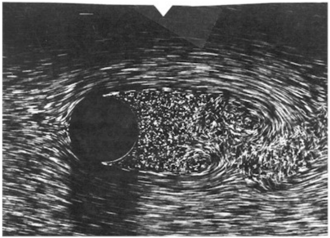


Figure: Cross-section of a flow past a cylinder (source: ONERA, France)

Goldstein singularity

- Formal computations of a solution by [Goldstein '48, Stewartson '58] (asymptotic expansion in well-chosen self-similar variables).
  Prediction: there exists a solution such that $\partial_Y u_{|Y=0}(x) \sim \sqrt{x^* - x}$ as $x \to x^*$.
  Heuristic argument by Landau giving the same separation rate.
- [D., Masmoudi, '18]: rigorous justification of the Goldstein singularity. Computation of an approximate solution, using modulation of variables techniques.
  Open problem: is $\sqrt{x^* - x}$ the "stable" separation rate?
- **Why "singularity"?**
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- In this case, "generically", recirculation causes separation.

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Open problems for the stationary case

- Remove Goldstein singularity by adding corrector terms in the equation, coming from the coupling with the outer flow (triple deck system?);
- Construct solutions with recirculation.

Justification of the Prandtl Ansatz

Overall idea: far from the separation point, as long as there is no re-circulation, the Prandtl Ansatz can be justified.

- [Guo& Nguyen, '17]: Navier-Stokes system above a moving plate (non-zero boundary condition), later extended by [Iyer];
- [Gérard-Varet& Maekawa, '18]: main order term in Prandtl is a shear flow;
- [Guo& Iyer, '18]: main order term in Prandtl is the Blasius boundary layer (self-similar solution).

All works rely on new coercivity estimates for the Rayleigh operator $R[\varphi] = U_s(\partial_Y^2 - k^2)\varphi - U_s''\varphi$ (in the case of a shear flow), and on some additional estimates: estimates on v in [GN17], estimates for the Airy operator in [GVM18], trace estimates in [GI18].

Remark: interestingly, all works except [Iyer] work in a domain of small size in x... Actual or technical limitation?

Plan

A reminder...

Time-dependent Prandtl equation (P):

$$\begin{aligned} \partial_t u + u\partial_x u + v\partial_Y u - \partial_{YY} u = -\frac{\partial p^E}{\partial x}(t,x,0) \\ \partial_x u + \partial_Y v = 0, \\ \mathbf{u}_{|Y=0} = 0, \ \lim_{Y\to\infty} u(x,Y) = u_\infty(t,x) := u^E(t,x,0), \\ u_{|t=0} = u_{ini}. \end{aligned}$$

$\sim$ (Degenerate) heat equation $\partial_t u - \partial_{YY} u$
$+$ local transport term $u\partial_x u$
$+$ non-local transport term with loss of one derivative $v\partial_Y u = -\int_0^Y u_x$.

Plan

Well-posedness in high regularity settings

Theorem [Sammartino& Caflisch, '98]: *Let u_{ini} be analytic in x with Sobolev regularity in Y. Then there exists a time $T_0 > 0$ such that a solution of the Prandtl system* (P) *exists on* $(0, T_0)$*. Furthermore, on the existence time of the solution, the Prandtl Ansatz holds true.*

Idea of the proof: use of Cauchy-Kowalevskaya theorem, after filtering out the heat semi-group.

Extensions: [Kukavica& Vicol, '13; Gérard-Varet& Masmoudi, '14] WP results for data that belong to Gevrey spaces with Gevrey regularity > 1. Use of clever non-linear cancellations to go above Gevrey regularity 1 (analytic functions).

[Maekawa, '14] When the initial vorticity $\omega^\nu_{ini} = \partial_y u^\nu_{ini} - \partial_x v^\nu_{ini}$ is supported far from the wall $y = 0$, the Prandtl solution exists on an interval of size $O(1)$ and the Prandtl Ansatz can be justified.

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Monotone setting

Theorem [Oleinik, '63-'66]: *If u_{ini} is such that $\partial_Y u_{ini}(x, Y) > 0$ for $Y > 0$ (monotonicity in Y), then existence of a local solution in Sobolev spaces.*

Proof relies on a nonlinear change of variables (Crocco transform: new vertical variable is u, new unknown is $\partial_Y u$.)

[Masmoudi & Wong, '15; Alexandre, Wang, Xu & Yang, '15] Proof of the same result by using energy estimates and non linear cancellations only (no change of variables).

Relies on estimates for the quantity

$$\omega - \frac{\partial_Y \omega}{\omega} u,$$

where $\omega := \partial_Y u$ (vorticity).

In this setting, the validity of the Prandtl Ansatz has been proved [Gérard-Varet, Maekawa& Masmoudi, '16], in the Gevrey setting, for concave shear flow boundary layers.

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Singularity formation in Sobolev spaces

- [E& Engquist, '97] For suitable initial data, satisfying $u_{ini}(0, y) = 0$ for all $y > 0$, proof of blow-up in Sobolev spaces by a virial type method (look for energy inequalities on the quantity $\partial_x u(t, 0, y)$).
- Later extended by [Kukavica, Vicol, Wang, '15]

Justification of the van Dommelen-Shen singularity.

Prandtl instabilities in Sobolev spaces

Starting point: consider a shear flow $(U_s(Y), 0)$, and the linearized Prandtl equation around it

$$\begin{aligned} \partial_t u + U_s \partial_x u + v \partial_Y U_s - \partial_{YY} u = 0, \\ \partial_x u + \partial_Y v = 0, \\ u_{|Y=0} = v_{|Y=0} = 0, \ \lim_{Y \to \infty} u(t,x,Y) = 0. \end{aligned} \tag{LP}$$

Look for spectral instabilities of the above system. The well-posedness results in the monotonic case suggest that no instability should occur if U_s is monotone.

Theorem [Gérard-Varet& Dormy, '10] *Let $(U_s(Y,0))$ be a shear flow such that U_s has a non-degenerate critical point. Then*

- *There exist approximate solutions whose k-th Fourier mode grows like $\exp(\alpha\sqrt{k}t)$ for some $\alpha > 0$;*
- *As a consequence, (LP) is ill-posed in Sobolev spaces.*

Former description (at a formal level) in [Cowley et al., '84].

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Nature of the instability in [Cowley; Gérard-Varet&Dormy]

Eq. (LP) has cst. coeff. in x → Fourier in x, t →ODE in Y.
Look for an instability → high frequency analysis in space&time.
Asymptotic expansion: close to a non-degenerate critical point a, the solution looks like

$$v^P(t,x,Y) \simeq \exp(ik(\omega t+x)) \left(\underbrace{v_a(Y)}_{\text{inviscid sol.}} + \underbrace{\epsilon^{1/2}\tau \mathbf{1}_{y>a} + \epsilon^{1/2}\tau V\left(\frac{y-a}{\epsilon^{1/4}}\right)}_{\text{viscous correction}} \right)$$

where $\epsilon := 1/|k| \ll 1$, $\omega = -U_s(a) + \epsilon^{1/2}\tau$, where $\tau \in \mathbb{C}$ is such that $\Im(\tau) < 0$.
Conclusion: the k-th mode grows like $\exp(|\Im(\tau)|\sqrt{|k|}t)$.
Remark: Viscosity induced instability.

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Interactive boundary layer models

Intuition: [Catherall& Mangler; Le Balleur; Carter; Veldman...]
At the point where a singularity is formed in the Prandtl system and the expansion ceases to be valid, the coupling with the interior flow must be considered at a higher order in ν, with potential stabilizing effects.
Cornerstone: notion of blowing velocity/displacement thickness: note that

$$v^P(x,Y) = -\int_0^Y u^P_x = -Y\partial_x u_\infty - \underbrace{\partial_x \int_0^Y (u^P - u_\infty)}_{=\text{``blowing velocity''}}.$$

Interactive boundary layer model: couple the Euler and the boundary layer systems by prescribing the following coupling condition:

$$v^E(t,x,0) = \sqrt{\nu}\partial_x \int_0^\infty (u_\infty - u^P(t,x,Y))\, dY.$$

Instabilities for the IBL system

Unfortunately, the linearized IBL system has even worse properties than Prandtl...

Theorem [D., Dietert, Gérard-Varet, Marbach, '17]

- *For any monotone shear flow U_s, there exist solutions of the linearized IBL system around U_s whose k-th mode grows like* $\exp(\alpha\nu^{3/4}k^2t)$ *in the regime* $|k| \gg \nu^{-3/4}$.
- *If U_s is monotone and $U_s''(0) > 0$, there exist solutions growing like* $\exp(\alpha\nu|k|^3t)$, *in the regime* $\nu^{-1/3} \ll |k| \ll \nu^{-1/2}$.

Remark: profiles are stable for Prandtl (monotone). Instabilities are much stronger than in the Prandtl case, and also stronger than Tollmien Schlichting instabilities.

Invalidity of the Prandtl Ansatz - 1

Starting point: Look at solution of the Navier-Stokes system with viscosity ν and initial data close to $(U_s(y/\sqrt{\nu}), 0)$.

Question: does the solution of the Navier-Stokes system remain close to $(e^{t\Delta} U_s)(y/\sqrt{\nu})$?

Answer: generically, no...

More precisely:

Theorem [Grenier, Guo, Nguyen, '16]:

- *If the profile U_s is unstable for the Rayleigh equation, there are modal solutions of the linearized NS system, of spatial frequency $\sim \nu^{-3/8}$ that grow like $\exp(ct\nu^{-1/4})$ (Tollmien-Schlichting waves);*
- *Similar result (in a possibly different regime) for profiles that are stable for the Rayleigh equation!*

Scheme of proof

Look for a solution of the linearized Navier-Stokes system in the form

$$\mathbf{u}^\nu = \nabla^\perp \psi^\nu, \text{ where } \psi^\nu(t,x,y) = \phi\left(\frac{y}{\sqrt{\nu}}\right) \exp\left(\frac{ik}{\sqrt{\nu}}(x-\omega t)\right).$$

Then ϕ solves the Orr-Sommerfeld equation:

$$(U_s - \omega)(\partial_Y^2 - k^2)\phi - U_s''\phi - \frac{\sqrt{\nu}}{ik}(\partial_Y^2 - k^2)^2\phi = 0.$$

• $\nu = 0$: Rayleigh equation (involved in stability of Euler).
Instability criteria: Rayleigh ($\exists$ inflexion point), Fjørtoft.
• If U_s is unstable for Rayleigh, construction of an approximate solution starting from an inviscid unstable mode and adding a viscous correction: sublayer of size $\nu^{3/4}$ within the boundary layer of size $\sqrt{\nu}$.
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Scheme of proof

Look for a solution of the linearized Navier-Stokes system in the form

$$\mathbf{u}^\nu = \nabla^\perp \psi^\nu, \text{ where } \psi^\nu(t,x,y) = \phi\left(\frac{y}{\sqrt{\nu}}\right)\exp\left(\frac{ik}{\sqrt{\nu}}(x-\omega t)\right).$$

Then ϕ solves the Orr-Sommerfeld equation:

$$(U_s - \omega)(\partial_Y^2 - k^2)\phi - U_s''\phi - \frac{\sqrt{\nu}}{ik}(\partial_Y^2 - k^2)^2\phi = 0.$$

• $\nu = 0$: Rayleigh equation (involved in stability of Euler).
Instability criteria: Rayleigh ($\exists$ inflexion point), Fjørtoft.
• If U_s is unstable for Rayleigh, construction of an approximate solution starting from an inviscid unstable mode and adding a viscous correction: sublayer of size $\nu^{3/4}$ within the boundary layer of size $\sqrt{\nu}$.
• For a stable mode, the construction is similar (but more complicated!)

Invalidity of the Prandtl Ansatz - 2

As a consequence of the previous construction, one obtains:

Theorem [Grenier '00; Grenier& Nguyen '18]: *There exists a solution of the Navier-Stokes system* $(U_s(y/\sqrt{\nu}), 0)$ *with source term* F^ν*, with the following properties: for any* N, s *(large), there exists* $\delta_0 > 0$*,* $c_0 > 0$*, and a solution* $\mathbf{u}^\nu$ *of NS with source term* f^ν*, such that:*

- $\|\mathbf{u}^\nu(t=0) - (U(\cdot/\sqrt{\nu}), 0)\|_{H^s} \leq \nu^N$;
- $\|f^\nu - F^\nu\|_{L^\infty([0,T^\nu],H^s)} \leq \nu^N$;
- $\|\mathbf{u}^\nu(t = T^\nu) - (U(\cdot/\sqrt{\nu}), 0)\|_{L^\infty} \geq \delta_0$, *with* $T^\nu \sim C_0\sqrt{\nu}|\ln \nu|$.

Summary

• **Stationary case:** the only mathematical setting in which solutions are known up to now is the case of positive solutions. For such a setting, we have a good understanding of singularities close to the separation point, and we are able to justify the Ansatz far from the separation.

• **Time-dependent case:** WP in high regularity settings and for monotone data.

In the non-monotone case, creation of vorticity close to the wall, that destabilizes the boundary layer. Strong instabilities in Sobolev spaces; the boundary layer Ansatz fails.

Conclusion

• Small scale structures (both in x AND y) appear close to the wall in general (cf. instabilities).

• The boundary layer Ansatz should be replaced by something else, accounting for small scale vortices. But... what ?

Thank you for your attention !